行政院國家科學委員會專題研究計畫成果報告
非線性取樣系統 H^∞ 控制器設計
On H^∞ control for nonlinear sampled-data systems

計畫編號:NSC-89-2213-E-019-024-
執行期限:89年8月1日至90年7月31日
主持人:容志輝 國立台灣海洋大學電機系
Email:yung@mail.ntou.edu.tw
參與計畫研究生:楊東峰,黎燕芳

摘要

本計畫主要研究一般性nonaffine非線性取樣系統 H^∞ 控制器之參數化問題.吾人推導一族 H^∞ 控制器之狀態空間表示式.此族 H^∞ 控制器具有離散時間系統串接一廣義持住函數之架構.

關鍵詞:H^∞ 控制,取樣系統, nonaffine 非線性系統,控制化參數.

Abstract

This project studies the H^∞ control problem for more general nonaffine nonlinear systems with sampled measurements. Sufficient conditions for existence of a family of H^∞ controllers built from sampled measurements are presented. State-space formulas for such a family of H^∞ controllers are also derived. It is proven that the family of controllers thus obtained has the structure of a discrete-time system followed by a generalized hold function. The results extend recent achievements in the literature.

Keywords: H^∞ control, sampled-data systems, nonaffine nonlinear systems, controller parametrization.

1. Introduction

Recently, much attention has been given to the extensions of the results of linear H^∞ control theory [5] [6] to nonlinear settings; see, e.g., [1]-[4], [8]-[13], and [17]-[22]. In particular, Van der Schaft [17] has shown that the solution to the nonlinear H^∞ state feedback control problem for affine nonlinear systems can be obtained by solving one Hamilton-Jacobi equation, which is the nonlinear version of the Riccati equation considered in the corresponding linear H^∞ control theory (see [5]). On the other hand, Ball et al. [4] and Isidori [9]have presented sufficient (or necessary) conditions based on two Hamilton-Jacobi equations (or inequalities) for the solution to the nonlinear H^∞ control problem for affine nonlinear systems (or W-input affine nonlinear systems [4]) in the case of output feedback. Furthermore, Isidori and Kang [8] and Van der schaft [19] have studied the H^∞ control problem for general nonaffine nonlinear systems. Moreover, Lin and Byrnes (see [11] and [12]) have obtained some corresponding results for discrete-time nonlinear systems.

In addition, in the work of Astolfi [1], Lu and Doyle [13], and Yung *et al.* [21], state-space formulas have been given for a family of H^∞ controllers for an affine nonlinear system via output feedback; a family of H^∞ controllers for an affine nonlinear system via state feedback has also been obtained in [22]. Furthermore, Lin and Byrnes [11] have presented a parameterization of a family of nonlinear H^∞ static state feedback controllers for discrete-time affine nonlinear systems.

In a recent monograph [20], among many other important contributions, Van der Schaft has treated a number of issues related to necessary conditions for solutions to exist for more general nonaffine nonlinear systems modeled by equations in which, in contrast to [8], the penalty output z (respectively, the measured output y) is not necessarily independent of the exogenous input w (respectively, the control input u). The results obtained in [20] extend the corresponding ones in [4]. Furthermore, Yung *et al.* [23] have presented a sufficient condition for the existence of a solution to the problem and construct a feedback law solving the problem via state feedback as well as output feedback. They have also derived state-space formulas for a family of H^∞ controllers via output feedback as well as state feedback.

Most recently, Suzuki *et al.* [16] and Guillard [7] have considered the H^∞ control problem for affine nonlinear systems with sampled measurements. The proposed approach by Suzuki *et al.* [16] is based on a certainty equivalence principle as used in [10] with a constant injection gain. On the other hand, the approach used in [7] allows the injection gain to be a nonlinear function of estimate.

This project continues this line of research to study the H^∞ control problem for more general nonaffine nonlinear systems (as in [23]) with sampled measurements. More precisely, sufficient conditions are presented for existence of a family of H^∞ controllers built from sampled measurements . State-space formulas for such a family of H^∞ controllers are also derived. It is proven that the family of controllers thus obtained has the structure of a discrete-time system followed by a generalized hold function. The results extend recent achievements [7] [8] [16] [23] in the literature.

2. Problem Formulation and Preliminaries

consider a smooth nonaffine nonlinear system described by the state equations

$$\begin{array}{ll} \dot{x} = F(x,w,u) & x(0)=0 \\ z = Z(x,w,u) & \end{array} \qquad (1)$$

where $x \in I\!R^n$ represents the state defined on a neighborhood of the origin, $u \in I\!R^m$ is the control input, $w \in I\!R^r$ represents a set of exogenous inputs, $z \in I\!R^s$ is the controlled variable. The measurements $y \in I\!R^p$ are assumed to be available at sampling instants ih, i.e.,

$$y(ih) = Y(x(ih), v(ih)) \qquad i \geq 1 \qquad (2)$$

where h is a fixed sampling period and the variable $v \in I\!R^p$ denotes the measurement noise. It is assumed throughout that $F(0,0,0)=0$, $Z(0,0,0)=0$ and $Y(0,0)=0$. In this project, we restrict ourselves to the consideration of systems satisfying the following assumptions (see also [8] and [23]).

Assumption A1: The matrix D_{12} has rank m and the matrix $D_{11}^T D_{11} - \gamma^2 I$ is negative definite, where $D_{12} = (\frac{\partial Z}{\partial u})_{(x,w,u)=(0,0,0)}$ and $D_{11} = (\frac{\partial Z}{\partial w})_{(x,w,u)=(0,0,0)}$.

Assumption A2: The matrix $D_{21} = (\frac{\partial Y}{\partial v})_{(x,v)=(0,0)}$ has full row rank.

The H^∞ control problem under consideration in this project is concerned with constructing a controller via the sampled measurements (2) such that the resulting closed-loop system has a locally asymptotically stable equilibrium at the origin and has L^2-gain $\leq \gamma$. Here the closed-loop L^2-gain takes into account both the continuous-time disturbance w and the discrete-time disturbance v as follows: Suppose that the closed-loop system has a locally asymptotically stable equilibrium at the origin. Then the closed-loop system is said to have L^2-gain $\leq \gamma$ if the inequality

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \left[\int_0^T \|w(t)\|^2 dt + \sum_{k=0}^{\lfloor T/h \rfloor} \|v(ih)\|^2 \right] \qquad (3)$$

holds for all $T \geq 0$, all w such that $\|w(t)\| < \varepsilon \, \forall t \geq 0$, and all v such that $\|v(ih)\| < \delta \, \forall i \geq 1$, where ε and δ are two arbitrary small positive numbers.

3. Main Results

3.1 Parameterization of a Family of Nonaffine Nonlinear H^∞ Control Under Sampled Measurements

The family of controllers to be considered is described by the dynamic equation of the form

$$\begin{array}{rcl} \dot{\xi} & = & F(\xi, \alpha_1(\xi), \alpha_2(\xi) + c(\eta)) + \hat{g}_1(\xi)c(\eta) \qquad t \neq ih \\ \xi(ih) & = & \xi(ih^-) + \hat{g}_0(\xi(ih^-))(y(ih) - Y(\xi(ih^-),0)) \\ \sigma(ih) & = & y(ih) - Y(\xi(ih^-),0) \\ \dot{\eta} & = & a(\eta) \qquad t \neq ih \\ \eta(ih) & = & \eta(ih^-) + b(\eta(ih^-))\sigma(ih) \\ u & = & \alpha_2(\xi) + c(\eta) \end{array} \qquad (4)$$

where ξ and η are defined on some neighborhoods of the origins in $I\!R^n$ and $I\!R^q$, respectively, with $\xi(0)=0$ and $\eta(0)=0$, $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are smooth functions with $a(0)=0$ and $c(0)=0$, $\hat{g}_0(\cdot)$ and $\hat{g}_1(\cdot)$ are C^k functions (for some $k \geq 1$) to be determined. $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ will be precise in the sequel. This is shown pictorially in Fig. 1, where

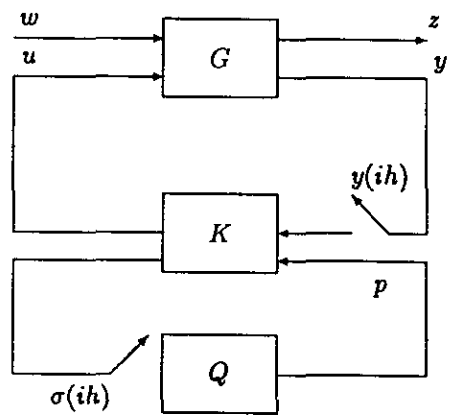

Fig. 1

$$G: \begin{cases} \dot{x} = F(x,w,u) \\ z = Z(x,w,u) \\ y(ih) = Y(x(ih), v(ih)) \end{cases}$$

$$K: \begin{cases} \dot{\xi} = F(\xi, \alpha_1(\xi), \alpha_2(\xi)+p) + \hat{g}_1(\xi)p \quad t \neq ih \\ \xi(ih) = \xi(ih^-) + \hat{g}_0(\xi(ih^-))(y(ih) - Y(\xi(ih^-),0)) \\ u = \alpha_2(\xi) + p \\ \sigma(ih) = y(ih) - Y(\xi(ih^-),0) \end{cases}$$

$$Q: \begin{cases} \dot{\eta} = a(\eta) \quad t \neq ih \\ \eta(ih) = \eta(ih^-) + b(\eta(ih^-))\sigma(ih) \\ p = c(\eta) \end{cases}$$

We want to show now how the functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$, $\hat{g}_0(\cdot)$ and $\hat{g}_1(\cdot)$ should be chosen such that the closed-loop system (4) is locally asymptotically stable and has L^2-gain$\leq \gamma$. To this end, we first reformulate the augmented system as

$$\begin{array}{rcll} \dot{X}^a &=& F^a(X^a, w) & t \neq ih \\ X^a(ih) &=& F^a_h(X^a(ih^-), v(ih^-)) & i \geq 1 \\ z &=& Z(x, w, \alpha_2(\xi) + c(\eta)) & \end{array} \qquad (5)$$

where $X^a = col(x, \xi, \eta)$,

$$F^a(X^a, w) = \begin{bmatrix} F(x,w,\alpha_2(\xi)+c(\eta)) \\ F(\xi,\alpha_1(\xi),\alpha_2(\xi)+c(\eta)) + \hat{g}_1(\xi)c(\eta) \\ a(\eta) \end{bmatrix},$$

and

$$F^a_h(X^a, v) = \begin{bmatrix} x \\ \xi + \hat{g}_0(\xi)(Y(x,v) - Y(\xi,0)) \\ \eta + b(\eta)(Y(x,v) - Y(\xi,0)) \end{bmatrix}.$$

A preliminary lemma will be needed in sequel.

Lemma 1

Consider *(1)* and *(2)*. Suppose that Assumption **A1** holds. Suppose that the following hypothesis holds.

(H1) There exists a smooth positive definite function $V(x)$, locally defined on a neighborhood of the origin in $\mathbb{R}^n$, such that the function

$$Y_1(x) = H(x, V_x^T(x), \alpha_1(x), \alpha_2(x)) \qquad (6)$$

is negative definite near $x = 0$, where the function $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}$ is defined on a neighborhood of $(x,p,w,u) = (0,0,0,0)$ as

$$H(x,p,w,u) = p^T F(x,w,u) + \|Z(x,w,u)\|^2 - \gamma^2\|w\|^2, \qquad (7)$$

$V_x(x)$ denotes the row vector $\frac{\partial V}{\partial x} \equiv [\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n}]$, $\alpha_1(x) = w^*(x, V_x^T(x))$, and $\alpha_2(x) = u^*(x, V_x^T(x))$, and where $w^*(x,p)$ and $u^*(x,p)$ are defined on a neighborhood of $(x,p) = (0,0)$, satisfying

$$\begin{array}{rcl} \frac{\partial H}{\partial w}(x,p,w^*(x,p),u^*(x,p)) &=& 0 \\ \frac{\partial H}{\partial u}(x,p,w^*(x,p),u^*(x,p)) &=& 0 \end{array}$$

with $w^*(0,0) = 0$ and $u^*(0,0) = 0$.

Let

$$\begin{array}{rcl} r_{11}(x) &=& \frac{1}{2}\left(\frac{\partial^2 H(x,V_x^T(x),w,u)}{\partial w^2}\right)_{w=\alpha_1(x),u=\alpha_2(x)} \\ r_{12}(x) &=& \frac{1}{2}\left(\frac{\partial^2 H(x,V_x^T(x),w,u)}{\partial u \partial w}\right)_{w=\alpha_1(x),u=\alpha_2(x)} \\ r_{21}(x) &=& \frac{1}{2}\left(\frac{\partial^2 H(x,V_x^T(x),w,u)}{\partial w \partial u}\right)_{w=\alpha_1(x),u=\alpha_2(x)} \\ r_{22}(x) &=& \frac{1}{2}\left(\frac{\partial^2 H(x,V_x^T(x),w,u)}{\partial u^2}\right)_{w=\alpha_1(x),u=\alpha_2(x)} \end{array}$$

with

$$\begin{bmatrix} r_{11}(0) & r_{12}(0) \\ r_{21}(0) & r_{22}(0) \end{bmatrix} = \begin{bmatrix} D_{11}^T D_{11} - \gamma^2 I & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix},$$

and set

$$R(x) = \begin{bmatrix} (1-\epsilon_1) r_{11}(x) & r_{12}(x) \\ r_{21}(x) & (1+\epsilon_2) r_{22}(x) \end{bmatrix},$$

where ϵ_1 and ϵ_2 are arbitrary but $0 < \epsilon_1 < 1$ and $\epsilon_2 > 0$. Suppose that the following condition also holds.

(a) There exists a real-valued function $M(X^a, t)$, locally defined on $\Psi_M \times \mathbb{R}^+$ with Ψ_M a neighborhood of the origin in $\mathbb{R}^{2n+q}$, which is C^3 with respect to X^a, h-periodic and piecewise differentiable with respect to t, vanishes at $X^a = col(x,x,0)$ for all $t \in [0,h]$ and is positive elsewhere, and is such that:

1) for all $t \in [0,h)$, the function

$$Y_2(X^a, t) = \tilde{H}(X^a, \tilde{w}(X^a,t), t) \qquad (8)$$

vanishes at $X^a = col(x,x,0)$ and is negative elsewhere, where the function $\tilde{H} : \mathbb{R}^{2n+q} \times \mathbb{R}^r \times \mathbb{R}^+ \to \mathbb{R}$ is defined on $\Phi_M \times \mathbb{R}^+$, with Φ_M a neighborhood of the origin in $\mathbb{R}^{2n+q} \times \mathbb{R}^r$, as

$$\tilde{H}(X^a,w,t) = M_t(X^a,t) + M_{X^a}(X^a,t)F^a(X^a,w) + \begin{bmatrix} w - \alpha_1(x) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(x) \end{bmatrix}^T R(x) \begin{bmatrix} w - \alpha_1(x) \\ \alpha_2(\xi) + c(\eta) - \alpha_2(x) \end{bmatrix}, \qquad (9)$$

and the function $\tilde{w}(\cdot,t)$, defined on a neighborhood of $X^a = 0$ for all t, is such that

$$\frac{\partial \tilde{H}(X^a,w,t)}{\partial w}\Big|_{w=\tilde{w}(X^a,t)} = 0, \qquad \tilde{w}(0,t) = 0.$$

and we set

$$\beta_1(x,\xi,\eta,t) = \tilde{w}(X^a,t)$$

2)

$$M_{X^aX^a}(0,h)\left(\frac{\partial F_h^a}{\partial v}\right)^T\Big|_{(X^a,v)=(0,0)}\left(\frac{\partial F_h^a}{\partial v}\right)\Big|_{(X^a,v)=(0,0)} - 2\gamma^2 I < 0 \quad (10)$$

3) the nonlinear difference inequality

$$\begin{aligned} Y_{2h}(X^a) &= M(F_h^a(X^a, v^*(X^a)), h) \\ &\quad - M(X^a, h^-) \\ &\quad - \gamma^2 v^{*T}(X^a) v^*(X^a) \le 0 \end{aligned} \quad (11)$$

is satisfied for all X^a near $X^a = 0$, where $v^*(X^a)$ is the unique solution with $v^*(0) = 0$ of the following implicit equation in v:

$$\frac{\partial M}{\partial \alpha}\Big|_{\alpha=F_h^a(X^a,v)}\left(\frac{\partial F_h^a}{\partial v}\right) - 2\gamma^2 v^T = 0 \quad (12)$$

Then the closed-loop system *(5)* is locally asymptotically stable around $X^a = 0$ and has L^2-gain $\le \gamma$.

The function $Y_2(X^a,t)$ thus obtained has $2n + q + 1$ independent variables and actually involves the undetermined quantities: $\hat{g}_0(\cdot)$, $\hat{g}_1(\cdot)$ and the free system data $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$. The next step is to show how the Condition (a) of Lemma 1 can be met.

Define a function

$$\begin{aligned} H_1(x, Q_x(x,t), w) &= Q_t(x,t) + Q_x(x,t)F(x,w,0) \\ &\quad + \begin{bmatrix} w - \alpha_1(x) \\ -\alpha_2(x) \end{bmatrix}^T \\ &\quad R(x)\begin{bmatrix} w - \alpha_1(x) \\ -\alpha_2(x) \end{bmatrix}. \end{aligned}$$

Then there exists a unique function $\hat{w}(x, Q_x^T(x,t))$, defined on a neighborhood of $(0,0)$, satisfies

$$\left(\frac{\partial H_1}{\partial w}\right)\Big|_{w=\hat{w}(x,Q_x^T(x,t))} = 0, \qquad \hat{w}(0,0) = 0,$$

by implicit function theorem since

$$\left(\frac{\partial^2 H_1}{\partial w^2}\right)\Big|_{(x,Q_x^T,w)=(0,0,0)} = 2(1-\epsilon_1)r_{11}(0) < 0$$

Define

$$\begin{aligned} \tilde{H}_1(\eta, K_\eta^T, \lambda) &= K_t(\eta,t) + K_\eta(\eta,t)a(\eta) \\ &\quad + \begin{bmatrix} \lambda \\ c(\eta) \end{bmatrix}^T \\ &\quad \begin{bmatrix} (1-\epsilon_1)r_{11}(0) & r_{12}(0) \\ r_{21}(0) & (1+\epsilon_2)r_{22}(0) \end{bmatrix} \\ &\quad \begin{bmatrix} \lambda \\ c(\eta) \end{bmatrix} \end{aligned}$$

Then, by the implicit function theorem again, there exists a function $\bar{\lambda}(\eta, K_\eta^T(\eta,t))$, defined on a neighborhood of $(0,0)$, satisfies

$$\left(\frac{\partial \tilde{H}_1}{\partial \lambda}\right)\Big|_{\lambda=\bar{\lambda}(\eta,K_\eta^T(\eta,t))} = 0, \qquad \bar{\lambda}(0,0) = 0.$$

Set

$$\alpha_4(\eta,t) = \bar{\lambda}(\eta, K_\eta^T(\eta,t)).$$

Theorem 1

Consider (1) *and* (4). *Suppose the function* $Y_v(0,0)Y_v^T(0,0) = I$, *and that Assumptions* **A1** and **A2** are satisfied. Suppose also Hypothesis **H1** and the following hold.

(a) There exists a smooth positive definite function $K(\eta,t)$, locally defined on $\Psi_K \times I\!R^+$, where Ψ_K is a neighborhood of the origin in $I\!R^q$, which is h-periodic, piecewise differentiable with respect to t, C^3 with respect to η, and has a nonsingular Hessian matrix $K_{\eta\eta}(0,h)$, such that the function $Y_3(\eta,t)$ is negative definite near $x = 0$ with a nonsingular Hessian matrix at $\eta = 0$ for all $t \in [0,h]$, where the function $Y_3(\eta,t)$ is defined as

$$\begin{aligned} Y_3(\eta,t) &= K_t(\eta,t) + K_\eta(\eta,t)a(\eta) \\ &\quad + \begin{bmatrix} \alpha_4(\eta,t) \\ c(\eta) \end{bmatrix}^T \\ &\quad \begin{bmatrix} (1-\epsilon_1)r_{11}(0) & r_{12}(0) \\ r_{21}(0) & (1+\epsilon_2)r_{22}(0) \end{bmatrix} \\ &\quad \begin{bmatrix} \alpha_4(\eta,t) \\ c(\eta) \end{bmatrix} \\ &\quad 0 \le t < h; \end{aligned}$$

$$Y_3(\eta,h) = K(\eta + b(\eta)Y(0, v^*(0,0,\eta)), h)$$
$$-K(\eta,h^-) + \frac{1}{2}\eta^T A\eta - \gamma^2 v^{*T}(0,0,\eta)v^*(0,0,\eta)$$

where

$$A = 2\gamma^2 v^{*T}_\eta(0) v^*_\eta(0)$$
$$-v^{*T}_\eta(0)Y_v^T(0,0)b^T(0)K_{\eta\eta}(0,h)[I + b(0)Y_v(0,0)v^*_\eta(0)]$$

(b) There exists a positive definite function $Q(x,t)$, locally defined on $\Psi_Q \times \mathbb{R}^+$, where Ψ_Q is a neighborhood of the origin in $\mathbb{R}^n$, which is h-periodic, piecewise differentiable with respect to t, C^3 with respect to x, and has a nonsingular Hessian matrix $Q_{xx}(0,h)$ satisfying

$$\begin{aligned}&(\hat{g}_0^T(0)Q_{xx}(0,h)\hat{g}_0(0)\\ &+b^T(0)K_{\eta\eta}(0,h)b(0)) - 2\gamma^2 I < 0\end{aligned} \qquad (13)$$

and is such that the function $S_1(x,t)$ is negative definite near $x=0$ with nonsingular Hessian matrix at $x=0$ for all $t \in [0,h]$, where the function $S_1(x,t)$ is defined as

$$\begin{aligned} S_1(x,t) &= Q_t(x,t) \\ &\quad +Q_x(x,t)F(x,\alpha_3(x,t),0) \\ &\quad + \begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix}^T \\ &\quad R(x)\begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix} \\ &\quad 0 \le t < h; \end{aligned} \qquad (14)$$

with

$$\begin{aligned}\alpha_3(x,t) &= \hat{w}(x, Q_x^T(x,t)) \\ S_1(x,h) &= Q(x,h) - Q(x,h^-) - \gamma^2 L^T(x)L(x).\end{aligned} \qquad (15)$$

with

$$L_x(x)\mid_{x=0} = Y_v^{-1}(0,0)Y_x(0,0) \qquad (16)$$

Then the function

$$M(X^a,t) = Q(x-\xi,t) + K(\eta,t)$$

satisfies the condition (a) of Lemma 1 if $\hat{g}_0$, $\hat{g}_1$ satisfy

$$Q_x(x,h)\hat{g}_0(x)Y_v(x,v^*(x,0,0)) = 2\gamma^2 L^T(x), \quad (17)$$

and

$$\begin{aligned}&-Q_x(x,t)\hat{g}_1(x) - 2(1+\epsilon_2)\alpha_2^T(x)r_{22}(x) \\ &\qquad +2\beta_1^T(x,0,0,t)r_{12}(x) = 0\end{aligned} \qquad (18)$$

3.2 Discretization of Controllers

In this subsection, a sufficient condition for the existence of $Q(x,t)$ satisfying the hypothese of Theorem 1 will be given . First, we observe that $Q(x,t)$ satisfies the equations as follows

$$\begin{aligned}&Q_t(x,t) + Q_x(x,t)F(x,\alpha_3(x,t),0) \\ &+\begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix}^T \\ &R(x)\begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix} = -\varepsilon_1(x) \\ &0 \le t < h;\end{aligned} \qquad (19)$$

and

$$Q(x,h) - Q(x,h^-) - \gamma^2 L^T(x)L(x) = -\varepsilon_2(x), \quad (20)$$

where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are two smooth positive definite function with nonsingular Hessian matrices $\varepsilon_{1xx}(0)$ and $\varepsilon_{2xx}(0)$. As a consequence, the problem comes down to finding a periodic solution $Q(x,t)$ of (19) and (20). Let us define a Hamiltonian function $H_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\begin{aligned} H_2(x,p) &= p^T F(x,\alpha_3(x,t),0) \\ &\quad + \begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix}^T \\ &\quad R(x)\begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix} + \varepsilon_1(x) \end{aligned} \qquad (21)$$

Then

$$H_{2x}(x,p) = F_w \cdot \alpha_{3x}p + F_x p + \frac{\partial}{\partial x}\Theta + \varepsilon_{1x}(x) \quad (22)$$

where

$$\Theta = \begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix}^T R(x)\begin{bmatrix} \alpha_3(x,t)-\alpha_1(x) \\ -\alpha_2(x)\end{bmatrix}$$

and

$$H_{2p}(x,p) = F(x,\alpha_3(x,t),0) \qquad (23)$$

Let $p = Q_x^T(x,t)$ and denote $x(t)$ the state of the system

$$\dot{x}(t) = H_{2p}^T(x(t), Q_x^T(x(t),t)) \qquad (24)$$

Then,

$$\begin{aligned}\dot{p}(t) &= Q_{tx}^T(x(t),t) \\ &= -H_{2x}^T(x(t), Q_x^T(x(t),t))\end{aligned} \qquad (25)$$

Furthermore, define the function $\Pi_1(x,p,t)$ and $\Pi_2(x,p,t)$ as

$$\begin{bmatrix}\Pi_1(x,p,t) \\ \Pi_2(x,p,t)\end{bmatrix} = e^{tL_{Ham}}(Id)\mid_{(x,p)}, \qquad (26)$$

where

$$e^{tL_{Ham}}(\cdot) = I + \sum_{i\ge 1} t^i L_{Ham}^i(\cdot)$$

is the Lie exponential series of $Ham(x,p)$ and $Ham(x,p)$ is defined by

$$Ham(x(t),p(t)) = \begin{bmatrix}\dot{x}(t) \\ \dot{p}(t)\end{bmatrix}. \qquad (27)$$

Obviously, $\Pi_1(x,p,t)$ and $\Pi_2(x,p,t)$ are the trajectories of the variables $x(t)$ and $p(t)$ respectively, which satisfy the equations of the system built on the Hamiltonian function (21). Thus $Q(x,t)$ must satisfy

$$\Pi_2(x, Q_x^T(x,0), t) = Q_x^T(\Pi_1(x, Q_x^T(x,0),t),t). \quad (28)$$

Then, we have the following result.

Theorem 2
Consider the system (1). Suppose that there exists a smooth positive definite function $\bar{Q}(x)$, locally defined on a neighborhood of the origin $x=0$ in $I\!R^n$, such that the following conditions hold:
(a)

$$\bar{Q}_x^T(\Pi_1(x, \bar{Q}_x^T(x) + 2\gamma^2 L_x^T(x)L(x) - {\varepsilon_2}_x^T(x), h))$$
$$= \Pi_2(x, \bar{Q}_x^T(x) + 2\gamma^2 L_x^T(x)L(x) - {\varepsilon_2}_x^T(x), h). \quad (29)$$

(b)

$$\frac{\partial \Pi_1}{\partial x}\Big|_{(x,p,t)=(0,0,t)} + \frac{\partial \Pi_1}{\partial p}\Big|_{(x,p,t)=(0,0,t)} \left[\bar{Q}_{xx}(0) + 2\gamma^2 L_x^T(0)L_x(0) - \varepsilon_{2xx}(0)\right] \quad (30)$$

is nonsingular for all $t \in [0,h]$. Consider the function $F_1(x,t)$ with $F_1(x,0) = x$ which is the unique solution of

$$\Pi_1(F_1, \bar{Q}_x^T(F_1) + 2\gamma^2 L_x^T(F_1)L(F_1) - {\varepsilon_2}_x^T(F_1), t) - x = 0 \quad (31)$$

satisfying $F_1(0,t) = 0$, for all $t \geq 0$.
Then, the function $Q(x,t)$ defined by

$$\begin{aligned} Q_x^T(x,t) &= \Pi_2(F_1(x,t), \Omega(x,t), t), \\ Q(0,t) &= 0, \qquad 0 \leq t < h, \end{aligned} \quad (32)$$

with

$$\Omega(x,t) = \bar{Q}_x^T(F_1(x,t)) + 2\gamma^2 L_x^T(F_1(x,t))L(F_1(x,t)) - {\varepsilon_2}_x^T(F_1(x,t)), \quad (34)$$

and

$$Q(x,h) = \bar{Q}(x) + \gamma^2 L^T(x)L(x) - \varepsilon_2(x). \qquad t = h \quad (35)$$

is a C^3 h-periodic solution of Theorem 1.

Under the conditions of Theorem 2, it follows immediately that

$$\bar{f}(x,\eta) = F(x, \alpha_1(x), \alpha_2(x) + c(\eta)) + \hat{g}_1(x)c(\eta)$$

$$F_1^*(\xi) = e^{tL_f}(Id)\,|_{\xi,\eta,t=h}$$
$$F_2^*(\eta) = e^{tL_*}(Id)\,|_{\eta,t=h}$$

where

$$\xi_i = \xi(ih), \qquad \xi_{i-1} = \xi((i-1)h)$$
$$\eta_i = \eta(ih), \qquad \eta_{i-1} = \eta((i-1)h)$$

A family of controllers with the structure of a discrete-time system followed by a generalized hold function can be constructed as follows:

$$\begin{aligned} \xi(ih) &= F_1^*(\xi_{i-1}) + \hat{g}_0(F_1^*(\xi_{i-1})) \\ &\quad [y(ih) - Y(F_1^*(\xi_{i-1}), 0)] \\ \eta(ih) &= F_2^*(\eta_{i-1}) + b(F_2^*(\eta_{i-1}))\sigma(ih) \\ u(t) &= u^*(e^{(t-ih)L_f}(Id)\,|_{\xi_i}) \\ &\quad + c(e^{(t-ih)L_*}(Id)\,|_{\eta_i}) \end{aligned} \quad (36)$$

where the injection gain $\hat{g}_0(\xi)$ satisfies

$$[\bar{Q}_x(\xi) + 2\gamma^2 L_x^T(\xi)L(\xi) - \varepsilon_{2x}(\xi)]\hat{g}_0(\xi) = 2\gamma^2 L^T(\xi) \quad (37)$$

4. Conclusions

The report has totally achieved the aim of the original proposal of the project. This is summarized as follows.

In the report, state-space formulas have been given for a family of H^∞ controllers for more general nonaffine nonlinear systems with sampled measurements. It has also been proven that the family of controllers thus obtained has the structure of a discrete-time system followed by a generalized hold function.

Due to space limitation, all proofs are omitted. One can consult [24] for more details.

References

[1] A. Astolfi, "Parametrization of output feedback controllers that satisfy an H^∞ norm bound," in *Proc. European Contr. Conf.*, Groningen, pp. 74-78, 1993.

[2] J.A. Ball and J.W. Helton, "Factorization of nonlinear systems: Towards a theory for nonlinear H^∞ control," in *Proc. 27th CDC*, pp. 2376-2381, 1988.

[3] J.A. Ball and J.W. Helton, "H^∞ control for nonlinear plants: connections with differential games," in *Proc. 28th CDC*, pp. 956-962, 1989.

[4] J.A. Ball, J.W. Helton and M.L. Walker, "H^∞ control for nonlinear systems with output feedback," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 546-559, 1993.

[5] J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis, "State-space solutions to standard H^2 and H^∞ control porblems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831-846, 1989.

[6] K. Glover and J.C. Doyle, "A state space approach to H^∞ optimal control," *Three Decades of Mathematical System Theory*, H.Nijmejier and H.Schumacher eds., LNCIS 135, Springer-Verlag, Berlin, pp. 179-218, 1989.

[7] H. Guillard, "On nonlinear H^∞ control under sampled measurements," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 880-885, 1997.

[8] A. Isidori and W. Kang, "H^∞ control via measurement feedback for general nonlinear systems," *IEEE Trans. Automat. Control*, vol. 40, pp. 466-472, 1995.

[9] A. Isidori, "H^∞ control via measurement feedback for affine nonlinear systems," *Int. J. of Robust and Nonlinear Control*, vol. 40, pp. 553-574, 1994.

[10] A. Isidori and A. Astolfi, "Disturbance attenuation and H^∞-control via measurement feedback in nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1283-1293, 1992.

[11] W. Lin and C.I. Byrnes, "Discrete-time nonlinear H^∞-control with measurement feedback," *Automatica*, vol. 31, no. 3, pp. 419-434, 1995.

[12] W. Lin and C.I. Byrnes, "H^∞-control of discrete-time nonlinear systems," *IEEE Trans. Automat. Control*, vol. 41, no. 4, pp. 494-510, 1996.

[13] W.M. Lu and J.C. Doyle, "H^∞ control of nonlinear system via output feedback: Controllers parametrization," *IEEE Trans. Automat. Control*, vol. 39, pp. 2517-2521, 1994.

[14] M.G. Safonov, D.J.N. Limebeer, and R.Y. Chiang, "Simplifying the H^∞ theory via loop shifting, matrix pencil and descriptor concepts," *Int. J. Control*, vol. 50, no. 6, pp. 2467-2488, 1990.

[15] W. Sun, K.M. Nagpal, and P.P. Khargonekar, "H^∞ control and filtering for sampled-data systems", *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1162-1175, 1993.

[16] S. Suzuki, A. Isidori, and T.J. Tarn, 1995, "H^∞ control of nonlinear systems with sampled measurement," *J. Math. Syst., Estimation Contr*, vol. 5, pp. 259-262.

[17] A.J. Van der Schaft, "On a state space approach to nonlinear H^∞ control," *Syst. Contr. Letters*, vol. 16, pp. 1-8, 1991.

[18] A.J. Van der Schaft, "L^2-gain analysis of nonlinear systems and nonlinear state feedback H^∞ control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 770-784, 1992.

[19] A.J. Van der Schaft, "Complements to nonlinear H^∞ optimal control by state feedback ," *IMA J. Math. Contr. Inf.*, vol. 9, pp. 245-254, 1992.

[20] A.J. Van der Schaft, *L^2-Gain and Passivity Techniques in Nonlinear Control*, Springer-Verlag, London, 1996.

[21] C.F. Yung, Y.P. Lin and F.B. Yeh, "A family of nonlinear H^∞ output feedback controllers," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 232-236, 1996.

[22] C.F. Yung, J.L. Wu and T.T. Lee, " Parameterization of nonlinear H^∞ state feedback controllers," *Automatica*, vol.33, pp. 1587-1590, 1997.

[23] C.F. Yung, J.L. Wu and T.T. Lee, " H^∞ control for more general nonlinear systems" *IEEE Trans. Automat. Contr.*, vol.43, pp. 4-7, 1998.

[24] C.F. Yung, Y.F. Li and Patrick Yang, " H^∞ control for general nonlinear systems with sampled measurements, " preprint. (submitted for publication)

參加2000年IEEE決策與控制(CDC)研討會

心得報告

撰寫人:容志輝
國立臺灣海洋大學電機工程學系
中華民國八十九年十二月二十日

(一)參加會議經過

每年 IEEE 所舉辦的 CDC(Conference on Decision and Control，決策與控制研討會)論文發表會是目前國際控制界最重要的全球性會議，來自世界各地控制領域的學者專家齊聚一堂，發表一年來研究心得，當今最先進的控制理論常會在此會議中發表。

CDC 會議大多在美國各大城市輪流舉行。除了少數幾年在其他國家，尤其是 1996 第一次在亞洲(日本神戶)舉行，今年則首次在南半球澳洲雪梨 12 月夏季舉行。十週之前，四年一度的奧林匹克運動會才在雪梨閉幕，兩週之後(2001 年 1 月 1 日)正好又是澳大利亞獨立建國一百週年紀念日。這一連串的大事，不禁讓人深刻感受到，澳大利亞充滿信心，懷抱熾熱的旺盛企圖心，邁向二十一世紀的希望遠景。

也許是拜雪梨之賜，今年投稿 CDC 的論文暴增，於是論文被接受的比例相對的劇減。

今年 CDC 仍然維持四天的議程，筆者的論文被安排在第三天早上發表。此篇論文"On the design of reduced - order H^{∞} controllers for nonaffine nonlinear systems" 主要探討對一般 nonaffine 非線性系統如何設計低階的 H^{∞}控制

器。建構 H^{∞}控制器需要解兩組代數 Riccati 方程式(線性情況)或兩組 Hamilton-Jacobi 不等式(非線性情況)。由此所設計的 H^{∞} 控制器其階數等於或高於廣義受控體(generalized plant)的階數。廣義受控體通常由真正的受控體(如馬達)加上一些迴路整形(loop-shaping)之頻域加權函數(frequency weighting function)，因此廣義受控體的階數往往變得很高，依此所得 H^{∞}控制器之階數亦相對地變得很高，在實際實現(implementation)時，將增加製作之成本及複雜性。

近兩年來，筆者致力於研究 H^{∞} 控制器之降階問題，得到一些突破性的發現。筆者發現線性 H^{∞} 控制器之階數可降至某代數 Riccati 方程式穩定解 W_{∞} 之秩(rank)。近來，筆者亦將此降階方法成功地推廣到 affine 非線性 H^{∞}控制問題，此乃發表在國際期刊第一個研究非線性 H^{∞}控制器降階問題的論文。(以上兩篇論文分別發表在 2000 年 Automatica Vol.36 及 2001 年 Automatica Vol.37)今年筆者在 CDC 發表的論文則將上述成果進一步推廣到更一般性 nonaffine 非線性系統 H^{∞}控制器之降階設計。處理 nonaffine 非線性系統之數學問題相當複雜。筆者論文主要貢獻在於

提出處理這些複雜數學背後的幾何意義另類思考。

(二)與會心得

觀察今年 CDC 安排的 Plenary Lectures 及各個 session 主題，可以發現國際上控制的潮流朝兩大方向：一是數理控制理論(mathematical control theory)中多年來懸而未決的 open problem 至今仍是一團謎.許多學者嘗試以新的數學解決這些問題，如線性時變系統的極點與零點的問題，據筆者知識所及，這些問題仍有待更多心血注入；二是控制與其他學門的整合。如控制理論應用在訊號處理，或通訊網路及光機電整合等先進科技。筆者相信這樣的結合在未來幾年必呈現雨後春筍之快速發展，預期唯此整合技術之研究方可培育相關高科技系統整合之人才。

另外值得一提的是大會在第二天下午特別安排一個遊覽整個雪梨海灣的海上巡禮，包括世界著名的雪梨歌劇院及雪梨海港大橋等盡收眼底。如此美麗的城市難怪每年吸引大量觀光客。筆者不禁想到台灣四面臨海，兼有百嶽雪嶺，並蘊藏豐富原始生態，我們是否盡心去維護這些天然資源呢？法國巴黎有美麗的塞納河，台北的基隆河及淡水河又如何呢？

(三)攜回資料名稱

1. 2000 年 CDC 研討會論文摘要一冊。

2. 2000 年 CDC 研討會論文全文 CD-ROM 一片。

(四)致謝

感謝國科會給予筆者赴會發表論文之補助，謹致由衷謝意。

参加2000 IEEE CDC發表論文之影本 pp.3043-3048

On the design of reduced-order H^∞ controllers for nonaffine nonlinear systems

Chee-Fai Yung [1] and Chau-ien Liu
Department of Electrical Engineering
National Taiwan Ocean University
Keelung 202, Taiwan, ROC
E-mail : yung@mail.ntou.edu.tw
Tel/Fax : +886-2-24626993

Abstract

Sufficient conditions are proposed for the existence of reduced-order (fixed-order) controllers solving the H^∞ control problem for nonaffine nonlinear systems. State-space formulas for such reduced-order H^∞ controllers are also derived in terms of the solutions to two standard Hamilton-Jacobi-Isaacs inequalities.

Keywords: H^∞ control, Controller reduction, Hamilton-Jacobi-Isaacs inequality, Nonaffine nonlinear systems

1 Introduction

It has been shown that full-order H^∞ controllers can be constructed from two algebraic Riccati equations for linear systems or two Hamilton-Jacobi-Isaacs inequalities for nonlinear systems. The controllers thus obtained have a state dimension not less than that of the generalized plant (Doyle *et al.*, 1989; Petersen *et al.*, 1991; Ball *et al.*, 1993; Isidori, 1994; Isidori and Kang, 1995; Lu and Doyle, 1994; Yung *et al.*, 1996; Yung *et al.*, 1998). Since the generalized plant is built from the physical plant and some weighting functions that are used to reflect performance and robustness requirements, the order of generalized plant may be very high. In this case, the full-order controllers may be of limited use in practical applications.

Recently, a number of papers have appeared that deal with reduced-order (or fixed-order) H^∞ controller design for linear systems (see, e.g., DeShetler and Ridgely, 1992; Gahinet and Apkarian, 1994; Gu *et al.*, 1993; Haddad and Bernstein, 1990; Hsu *et al.*, 1994; Hyland and Bernstein, 1984; Iwasaki and Skelton, 1993 and 1994; Juang *et al.*, 1996; Li and Chang, 1993; Pensar and Toivonen, 1993; Stoorvogel *et al.*, 1991; Sweriduk and Calise, 1993; Xin *et al.*, 1996; Yeh *et al.*., 1993; Yung, 2000).Most recently, the reduced-order H^∞ controller design problem for affine nonlinear systems has been addressed and extensively studied by Yung(1999). In terms of the two standard Hamilton-Jacobi-Isaacs inequalities (Isidori, 1994), sufficient conditions for the existence of reduced-order (fixed-order) nonlinear H^∞ controllers have been derived, and state-space formulas for such reduced-order nonlinear H^∞ controllers have also been provided (Yung 1999).

The purpose of the present paper is to extend the results of (Yung, 1999) to general nonaffine nonlinear systems. Sufficient conditions will be established for the existence of reduced-order (fixed-order) controllers solving the H^∞ controll problem for nonaffine nonlinear systems. State-space formulas for such reduced-order H^∞ controllers will also be derived.

2 Problem Formulation and Preliminaries

Consider a smooth (i.e. C^∞) nonaffine nonlinear system described by the state equations

$$\dot{x} = X(x,w,u) \quad (1a)$$
$$z = Z(x,w,u) \quad (1b)$$
$$y = Y(x,w,u), \quad (1c)$$

where x represents the state defined on a neighborhood of the origin in $\mathbb{R}^n$, $u \in \mathbb{R}^{m_2}$ is the control input, $w \in \mathbb{R}^{m_1}$ represents a set of exogenous inputs which includes disturbances to be rejected and/or reference commands to be tracked, $z \in \mathbb{R}^{p_1}$ is the controlled variable, and $y \in \mathbb{R}^{p_2}$ is the measured variable. It is assumed throghout that $X(0,0,0) = 0, Z(0,0,0) = 0$ and $Y(0,0,0) = 0$. In this paper, we restrict ourselves to the consideration of systems satisfying the following assumptions; see also Isidori and Kang(1995) and Yung *et al.*(1998).

[1] To whom all correspondence should be addressed

Assumption (A1): The matrices D_{12} has rank m_2 and the matrix $D_{11}^T D_{11} - \gamma^2 I$ is negative definite, where $D_{12} = (\frac{\partial Z}{\partial u})_{(x,w,u)=(0,0,0)}$ and $D_{11} = (\frac{\partial Z}{\partial w})_{(x,w,u)=(0,0,0)}$.

Assumption (A2): The matrix $D_{21} = (\frac{\partial Y}{\partial w})_{(x,w,u)=(0,0,0)}$ has rank p_2.

Our aim in this paper is to find a smooth reduced-order (fixed-order) output feedback controller of the form

$$\begin{aligned} \dot{\xi} &= \tilde{F}(\xi, y) \\ u &= H(\xi) \end{aligned} \tag{2}$$

where $\xi \in I\!R^r$ ($r \leq n$) is defined on a neighborhood of the origin, with $\tilde{F}(0,0) = 0$ and $H(0) = 0$, such that the resulting closed-loop system has a locally asymptotically stable equilibrium at the origin $(x,\xi)=(0,0)$, and has L^2-gain $\leq \gamma$, or equivalently, such that there exists a neighborhood of the origin $(x,\xi) = (0,0)$ such that for all $T > 0$ and for each input $w(\cdot) \in L^2[0,T]$, the state trajectory of the closed-loop system starting from the initial state $(x(0), \xi(0)) = (0,0)$ remains in the neighborhood for all $t \in [0,T]$, and the response $z(\cdot)$ of the closed-loop system satisfies

$$\int_0^T \| z(t) \|^2 \, dt \leq \gamma^2 \int_0^T \| w(t) \|^2 \, dt.$$

For details, see Van der Schaft (1992).

The following proposition follows immediatelly from the results of Isidori and Kang (1995), which provides an output feedback controller solving the problem in question.

Proposition 1 *Consider system* (1) *and suppose that Assumptions* **(A1)-(A2)** *are satisfied. Suppose that the following hypotheses hold.*

(H1) *There exists a smooth, positive definite function $V(x)$, locally defined on a neighborhood of the origin in $I\!R^n$, such that the function*

$$Y_1(x) = L(x, (\tfrac{dV}{dx})^T(x), \alpha_1(x), \alpha_2(x)) \tag{3}$$

is negative definite near $x = 0$, where the function $L : I\!R^n \times I\!R^n \times I\!R^{m_1} \times I\!R^{m_2} \to I\!R$ is defined on a neighborhood of $(x,p,w,u) = (0,0,0,0)$ as

$$L(x,p,w,u) = p^T X(x,w,u) + \|Z(x,w,u)\|^2 - \gamma^2 \|w\|^2,$$

where $\alpha_1(x) = w^*(x, (\frac{dV}{dx})^T(x))$, and $\alpha_2(x) = u^*(x, (\frac{dV}{dx})^T(x))$, and where $w^*(x,p)$ and $u^*(x,p)$ are defined on a neighborhood of $(x,p) = (0,0)$, satisfying

$$\begin{aligned} \frac{\partial L}{\partial w}(x,p,w^*(x,p),u^*(x,p)) &= 0 \\ \frac{\partial L}{\partial u}(x,p,w^*(x,p),u^*(x,p)) &= 0 \end{aligned}$$

with $w^*(0,0) = 0$ and $u^*(0,0) = 0$.

(H2) *There exists a smooth, positive definite function $W(x)$, locally defined on a neighborhood of $x = 0$, such that the function*

$$Y_2(x) = K(x, (\tfrac{dW}{dx})^T(x), \hat{w}(x, (\tfrac{dW}{dx})^T(x), \hat{y}(x, (\tfrac{dW}{dx})^T(x))), \hat{y}(x, (\tfrac{dW}{dx})^T(x))) - Y_1(x)$$

is negative definite near $x = 0$, and its Hessian matrix is nonsingular at $x = 0$, where the function $K : I\!R^n \times I\!R^n \times I\!R^{m_1} \times I\!R^{p_2} \to I\!R$ is defined on a neighborhood of $(x,p,w,y) = (0,0,0,0)$ as

$$K(x,p,w,y) = p^T X(x,w,0) - y^T Y(x,w,0) + \|Z(x,w,0)\|^2 - \gamma^2 \|w\|^2,$$

the function $\hat{w}(x,p,y)$, defined on a neighborhood of $(0,0,0)$, satisfies

$$(\frac{\partial K(x,p,w,y)}{\partial w})_{w=\hat{w}(x,p,y)} = 0, \quad \hat{w}(0,0,0) = 0,$$

and the function $\hat{y}(x,p)$, defined on a neighborhood of $(0,0)$, is such that

$$(\frac{\partial K(x,p,\hat{w}(x,p,y),y)}{\partial y})_{y=\hat{y}(x,p)} = 0, \quad \hat{y}(0,0) = 0.$$

Then if $W(x) - V(x) > 0$ for all $x \neq 0$, and if the equation

$$(\frac{dW}{dx}(x) - \frac{dV}{dx}(x))\hat{G}(x) = \hat{y}^T(x, (\frac{dW}{dx})^T(x))$$

has a smooth solution $\hat{G}(x)$ near $x = 0$, the nonlinear H^∞ output feedback control problem is solved by the output feedback

$$\begin{aligned} \dot{\hat{x}} &= \hat{F}(\hat{x}) + \hat{G}(\hat{x})y \\ u &= \hat{H}(\hat{x}) \end{aligned} \tag{4}$$

where $\hat{x} \in I\!R^n$ is defined on a neighborhood of the origin,

$$\hat{F}(\hat{x}) = X(\hat{x}, \alpha_1(\hat{x}), \alpha_2(\hat{x})) - \hat{G}(\hat{x})Y(\hat{x}, \alpha_1(\hat{x}), \alpha_2(\hat{x})),$$

and

$$\hat{H}(\hat{x}) = \alpha_2(\hat{x}).$$

3 Main Results

In this section, we will propose a reduced-order controller of the form (2) that locally asymptotically stabilizes the resulting closed-loop system and renders its L^2-gain $\leq \gamma$. For this purpose, suppose that the hypotheses (H1) and (H2) of Proposition 1 hold. We also assume that there exists a smooth function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^r$ defined around the origin $x = 0$ in $\mathbb{R}^n$ with $\phi(0) = 0$ and rank$\frac{d\phi}{dx}(0) = r$. The rank condition implies that the restriction of ϕ to some neighborhood Ω of $x = 0$ is a surjection. Then we make a change of variables

$$\hat{\xi} = -\phi(x) + \xi \tag{5}$$

where $\hat{\xi} \in \mathbb{R}^r$ and $\xi \in \mathbb{R}^r$ are defined on a neighborhood of the origin. In terms of these variables the resulting closed-loop system is

$$\begin{array}{rcl} \dot{x_e} & = & F_e(x_e, w) \\ z & = & H_e(x_e, w) \end{array} \tag{6}$$

where $x_e = col(x, \hat{\xi})$,

$$F_e(x_e, w) =$$

$$\begin{bmatrix} X(x,w,H(\hat{\xi}+\phi(x))) \\ -\frac{d\phi}{dx}(x)X(x,w,H(\hat{\xi}+\phi(x)))+\bar{F}(\hat{\xi}+\phi(x),Y(x,w,H(\hat{\xi}+\phi(x))) \end{bmatrix},$$

and

$$H_e(x_e, w) = Z(x, w, H(\hat{\xi} + \phi(x))).$$

The problem of choosing a control law (2) in such a way that the L^2-gain of the closed-loop system (6) from the exogenous input w to the penalty output z is less than or equal to γ can be viewed as a game problem of rendering the so-called Hamiltonian function $J : \mathbb{R}^{(n+r)} \times \mathbb{R}^{(n+r)} \times \mathbb{R}^{m_1} \rightarrow \mathbb{R}$ defined as

$$J(x_e, p, w) = p^T F_e(x_e, w) + \|H_e(x_e, w)\|^2 - \gamma^2 \|w\|^2$$

nonpositive for each x_e and each (p, w). It is easy to see by the implicit function theorem that, on a neighborhood of the point $(x_e, p, w) = (0, 0, 0)$, the Hamiltonian function can be rewritten as

$$\begin{aligned} J(x_e, p, w) = {} & J(x_e, p, w^{**}(x_e, p)) + \|w - w^{**}(x_e, p)\|^2_{\beta(x)} \\ & + o(\|w - w^{**}(x_e, p)\|^3), \end{aligned} \tag{7}$$

where $w^{**}(x_e, p)$ is a smooth function, defined on a neighborhood of $(0, 0)$, satisfying

$$\frac{\partial J}{\partial w}(x_e, p, w^{**}(x_e, p)) = 0, w^{**}(0, 0) = 0, \tag{8}$$

and is such that $\beta(x) = \frac{1}{2}\frac{\partial^2 J}{\partial w^2}(x_e, p, w^{**}(x_e, p))$ with $\beta(0) = D_{11}^T D_{11} - \gamma^2 I < 0$. Here the notation $\|v\|^2_M$ stands for $v^T M v$.

A preliminary lemma will be needed in the sequel.

Lemma 2 *Consider* (1), (2) *and* (5). *Suppose that Assumptions* **(A1)** *and* **(A2)** *are satisfied. Suppose also that there exists a smooth, positive definite function* $P(x_e)$, *locally defined on a neighborhood of the origin in* $\mathbb{R}^{n+r}$, *such that the function* $J^*(x_e) = J^*(x, \hat{\xi}) := J(x_e, (\frac{dP}{dx_e})^T(x_e), \alpha_3(x_e))$ *is negative for all nonzero* x_e *around* $x_e = 0$, *where* $\alpha_3(x_e) := w^{**}(x_e, (\frac{dP}{dx_e})^T(x_e))$. *Then the controller* (2) *locally asymptotically stabilizes the resulting closed-loop system* (6) *and renders its* L^2-*gain* $\leq \gamma$.

Proof:

Consider the candidate Lyapunov function $P(x_e)$. It is easy to see that, along the trajectories of the closed-loop system,

$$J(x_e, (\frac{dP}{dx_e})^T(x_e), w) = \frac{dP}{dt} + \|z\|^2 - \gamma^2 \|w\|^2 \tag{9}$$

Setting $w = 0$ in the above equality shows that $\frac{dP}{dt}$ is negative definite near $x_e = 0$. This proves that the equilibrium $x_e = 0$ of the closed-loop system is locally asymptotically stable. Furthermore, since $J^*(x_e) \leq 0$ for all x_e near $x_e = 0$ by hypothesis, (7) and (9) together imply that

$$\frac{dP}{dt} + \|z\|^2 - \gamma^2 \|w\|^2 \leq 0$$

from which we conclude that the closed-loop system (6) has L^2-gain $\leq \gamma$. This completes the proof. □

The remaining question now is how the condition in Lemma 2 can be met; that is, how can we make the function $J^*(x, \hat{\xi})$ negative for all nonzero x_e near $x_e = 0$. Furthermore, we also want to characterize the unknown parameters in (2), namely $\bar{F}$ and H.

First, we apply Taylor expansion theorem to $J^*(x, \hat{\xi})$ around $\hat{\xi} = 0$, obtaining

$$J^*(x, \hat{\xi}) = J^*(x, 0) + \frac{\partial J^*}{\partial \hat{\xi}}(x, 0)\hat{\xi} + \frac{1}{2}\hat{\xi}^T \frac{\partial^2 J^*}{\partial \hat{\xi}^2}(x, 0)\hat{\xi} + h.o.t., \tag{10}$$

where "$h.o.t.$" means higher order terms. Then, we observe that in order to make $J^*(x, \hat{\xi}) \leq 0$ for all nonzero $(x, \hat{\xi})$, it suffices to have (a)$J^*(x, 0) \leq 0$, (b)$\frac{\partial J^*}{\partial \hat{\xi}}(x, 0) = 0$, and (c)$\frac{\partial^2 J^*}{\partial \hat{\xi}^2}(x, 0) < 0$. In what follows, we shall make it clear how the three conditions can be met.

We begin the discussion with Condition (a). Let $U : \mathbb{R}^r \rightarrow \mathbb{R}$ be a smooth, positive definite function, locally defined on a neighborhood of the origin. With $P(x, \hat{\xi}) = V(x) + U(\hat{\xi})$ and

$$H(\phi(x)) = \alpha_2(x), \tag{11}$$

and with (8) in mind, if we first differentiate $J(x_e, (\frac{dP}{dx_e})^T(x_e), w)$ with respect to w, then substitute w for $\alpha_3(x_e)$, and finally set $\hat{\xi} = 0$, we can obtain

$$\begin{aligned}
&\tfrac{dV}{dx}(x)\tfrac{\partial X}{\partial w}(x,\alpha_3(x,0),\alpha_2(x)) \\
&\quad +2Z^T(x,\alpha_3(x,0),\alpha_2(x))\tfrac{\partial Z}{\partial w}(x,\alpha_3(x,0),\alpha_2(x))-2\gamma^2\alpha_3^T(x,0) \\
&=\tfrac{\partial L}{\partial w}(x,(\tfrac{dV}{dx})^T(x),\alpha_3(x,0),\alpha_2(x)) \\
&=0.
\end{aligned}$$

Since $(\alpha_1(x), \alpha_2(x))$ is the unique pair satisfying

$$\frac{\partial L}{\partial w}(x, (\frac{dV}{dx})^T(x), \alpha_1(x), \alpha_2(x)) = 0$$

and

$$\frac{\partial L}{\partial u}(x, (\frac{dV}{dx})^T(x), \alpha_1(x), \alpha_2(x)) = 0,$$

(see Isidori and Kang, 1995), we conclude that $\alpha_3(x, 0) = \alpha_1(x)$. Then it is easy to show that

$$\begin{aligned}
J^*(x,0) &= \tfrac{dV}{dx}(x)X(x,\alpha_1(x),\alpha_2(x))+\|Z(x,\alpha_1(x),\alpha_2(x))\|^2-\gamma^2\|\alpha_1(x)\|^2 \\
&= Y_1(x)
\end{aligned}$$

which is negative for all nonzero x by hypothesis.

Next, consider Condition (b). It is easy to verify that

$$\begin{aligned}
&\tfrac{\partial J^*}{\partial \xi}(x,0) \\
&=[\tfrac{dV}{dx}(x)\tfrac{\partial X}{\partial w}(x,\alpha_1(x),\alpha_2(x)) \\
&\quad +2Z^T(x,\alpha_1(x),\alpha_2(x))\tfrac{\partial Z}{\partial w}(x,\alpha_1(x),\alpha_2(x))-2\gamma^2\alpha_1^T(x)]\tfrac{\partial \alpha_3}{\partial \xi}(x,0) \\
&\quad +[\tfrac{dV}{dx}(x)\tfrac{\partial X}{\partial u}(x,\alpha_1(x),\alpha_2(x)) \\
&\quad +2Z^T(x,\alpha_1(x),\alpha_2(x))\tfrac{\partial Z}{\partial u}(x,\alpha_1(x),\alpha_2(x))]\tfrac{dH}{d\xi}(\phi(x)) \\
&\quad +[\tilde{F}(\phi(x),Y(x,\alpha_1(x),\alpha_2(x)))-\tfrac{d\phi}{dx}(x)X(x,\alpha_1(x),\alpha_2(x))]^T\tfrac{d^2U}{d\xi^2}(0) \\
&=\tfrac{\partial L}{\partial w}(x,(\tfrac{dV}{dx})^T(x),\alpha_1(x),\alpha_2(x))\tfrac{\partial \alpha_3}{\partial \xi}(x,0) \\
&\quad +\tfrac{\partial L}{\partial u}(x,(\tfrac{dV}{dx})^T(x),\alpha_1(x),\alpha_2(x))\tfrac{dH}{d\xi}(\phi(x)) \\
&\quad +[\tilde{F}(\phi(x),Y(x,\alpha_1(x),\alpha_2(x)))-\tfrac{d\phi}{dx}(x)X(x,\alpha_1(x),\alpha_2(x))]^T\tfrac{d^2U}{d\xi^2}(0) \\
&=[\tilde{F}(\phi(x),Y(x,\alpha_1(x),\alpha_2(x)))-\tfrac{d\phi}{dx}(x)X(x,\alpha_1(x),\alpha_2(x))]^T\tfrac{d^2U}{d\xi^2}(0)
\end{aligned}$$

We now set

$$\tilde{F}(\phi(x), Y(x, \alpha_1(x), \alpha_2(x))) = \frac{d\phi}{dx}(x)X(x, \alpha_1(x), \alpha_2(x)). \tag{12}$$

Then $\frac{\partial J^*}{\partial \xi}(x, 0) = 0$.

It is possible to further simplify the expression (12) if we confine attention to an *affine* reduced-order controller, i.e., if $\tilde{F}(\xi, y)$ has the form

$$\tilde{F}(\xi, y) = F(\xi) + G(\xi)y \tag{13}$$

where F and G are some smooth functions, both locally defined on some neighborhood of the origin $\xi = 0$ in $\mathbb{R}^r$ with $F(0) = 0$. It follows from (12) and (13) that

$$\begin{aligned}
&F(\phi(x)) + G(\phi(x))Y(x, \alpha_1(x), \alpha_2(x)) = \\
&\qquad \tfrac{d\phi}{dx}(x)\hat{F}(x) + \tfrac{d\phi}{dx}(x)\hat{G}(x)Y(x, \alpha_1(x), \alpha_2(x)),
\end{aligned}$$

where $\hat{F}$ and $\hat{G}$ are defined as in Proposition 1. Then it suffices to choose

$$F(\phi(x)) = \frac{d\phi}{dx}(x)\hat{F}(x) \tag{14}$$

$$\text{and} \quad G(\phi(x)) = \frac{d\phi}{dx}(x)\hat{G}(x) \tag{15}$$

to render $\frac{\partial J^*}{\partial \xi}(x, 0) = 0$

Finally, we consider Condition (c). For our purposes we shall further make the following assumption: Suppose that ϕ and U satisfy

$$\frac{d\phi}{dx}(0)(\frac{d\phi}{dx})^T(0) = I \tag{16}$$

and

$$\frac{d^2U}{d\xi^2}(0)\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(0)(\frac{d^2W}{dx^2}(0) - \frac{d^2V}{dx^2}(0)) \tag{17}$$

Then it is straightforward but tedious to verify that

$$\frac{\partial^2 J^*}{\partial \hat{\xi}^2}(0, 0) = \frac{d\phi}{dx}(0)\frac{d^2Y_2}{dx^2}(0)(\frac{d\phi}{dx})^T(0).$$

In summary, since, by hypothesis, $Y_1(x)$ is negative definite and $\frac{d^2Y_2}{dx^2}(0)$ is also negative definite, this shows that $J^*(x, \hat{\xi})$ is negative for all nonzero x_e around $x_e = 0$. We thus have the following theorem, which is the main result of this paper.

Theorem 3 *Suppose that Assumptions* **(A1)-(A2)** *are satisfied and that Hypotheses* **(H1)** *and* **(H2)** *of Proposition 1 hold. Suppose that there exists a smooth function* $\phi : \mathbb{R}^n \to \mathbb{R}^r$, *locally defined on a neighborhood of the origin* $x = 0$ *in* $\mathbb{R}^n$, *satisfying* $\phi(0) = 0$ *and (16). Suppose also that there exists a smooth, positive definite function* U, *locally defined on a neighborhood of the origin* $\xi = 0$ *in* $\mathbb{R}^r$, *which satisfies (17). Then, if* F, G, *and* H *satisfy (14), (15) and respectively (11), the* r*-th order controller*

$$\begin{aligned}
\dot{\xi} &= F(\xi) + G(\xi)y \\
u &= H(\xi)
\end{aligned}$$

locally asymptotically stabilizes the resulting closed-loop system (6) *and renders its* L^2*-gain* $\leq \gamma$.

4 Conclusions

A method has been proposed for designing reduced-order H^∞ controllers of general nonaffine nonlinear systems. When the system is an affine nonlinear system, it can be shown that the results obtained in this paper are exactly reduced to the corresponding results given in Yung(1999).

5 Acknowledgement

The work was supported by the National Science Council of the Republic of China under Grant NSC-88-2115-M-019-001.

References

[1] Ball, J.A., J.W. Helton and M.L. Walker (1993). H^∞ control for nonlinear systems with output feedback, *IEEE Trans. Automat. Control*, **38**, 546-559.

[2] DeShetler, D.R., and D.B. Ridgely (1992). Reduced-order H_2 and H_∞ compensation via gradient techniques, *Proc. of the 31th IEEE CDC*, Tuscon, Arizona, U.S.A., 2262-2267.

[3] Doyle, J.C., K. Glover, P.P. Khargonekar, and B.A. Francis (1989). State-space solutions to standard H^2 and H^∞ control problems, *IEEE Trans. Autom. Control*, **AC-34**, 831-847.

[4] Gahinet, P., and P. Apkarian (1994). A linear matrix ineqality approach to H^∞ control, *Int. J Robust Nonlinear Control*, **4**, 421-448.

[5] Gu, D.W., B.W. Choi, and I. Postlethwaite (1993). Low-order H_∞ suboptimal controllers, *Proc. of the 12nd World Congress of IFAC*, Sydney, Australia, **3**, 347-350.

[6] Haddad, W.M., and D.S. Bernstein (1990). Generalized Riccati equations for the full and reduced-order mixed-norm H_2/H_∞ standard problem, *Systems and Control Letters*, **14**, 185-197.

[7] Hsu, C.S., X. Yu, H.H. Yeh, and S.S. Banda (1994). H^∞ compensator design with minimal-order observers, *IEEE Trans. Autom. Control*, **39**, 1679-1681.

[8] Hyland, D.C., and D.S. Bernstein (1984). The optimal projection equations for fixed-order dynamic compensation, *IEEE Trans. Autom. Control*, **29**, 1034-1037.

[9] Isidori, A. (1994). H^∞ control via measurement feedback for affine nonlinear systems, *Int. J. Robust Nonlinear Control*, **4**, 553-574.

[10] Isidori, A., and W. Kang (1995). H^∞ control via measurement feedback for general nonlinear systems, *IEEE Trans. Autom. Control*, **40**, 466-472.

[11] Iwasaki, T., and R.E. Skelton (1993). All low order H_∞ controllers with covariance upper bound, *Proc. of the ACC*, San Francisco, California, U.S.A., 2180-2184.

[12] Iwasaki, T., and R.E. Skelton (1994). All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas, *Automatica*, **30**, 1307-1317.

[13] Juang, J.C., H.H. Yeh, and S.S. Banda (1996). Observer-based compensators for fixed-order H_∞ control, *Int. J. Control*, **64**, 441-461.

[14] Li, X.P., and B.C. Chang (1993). A parameterization approach to reduced-order H_∞ controller design, *Proc. of the 32nd IEEE CDC*, San Antonio, Texas, U.S.A., 2909-2184.

[15] Lu, W.M., and J.C. Doyle (1994). H^∞ control of nonlinear system via output feedback: controller parameterization, *IEEE Trans. Automat. Control*, **39**, 2517-2521.

[16] Pensar, J.A., and H.T. Toivonen (1993). On the design of fixed-structure H_∞ optimal controllers, *Proc. of the 32nd IEEE CDC*, San Antonio, Texas, U.S.A., 668-673.

[17] Petersen, I.R., B.D.O. Anderson and E.A. Jonckhcere (1991). A first principles solution to the nonsigular H_∞ control problem, *Int. J. Robust Nonlinear Control*, **1**, 171-185.

[18] Stoorvogel, A.A., A. Saberi, and B.M. Chen (1991). A reduced-order observer-based controller design for H_∞ optimization, *Proc. of the AIAA Guidance, Navigation, and Control Conference*, New Orleans, Louisiana, U.S.A., 716-722.

[19] Sweriduk, D., and A.J. Calise (1993). Robust fixed-order dynamic compensation: a differential game approach, *Proc. of the IEEE Conference on Aerospace Control Systems*, West Lake Village, California, U.S.A., 458-462.

[20] Van der Schaft, A.J. (1992). L^2-gain analysis of nonlinear systems and nonlinear state feedback H^∞ control, *IEEE Trans. Automat. Control*, **37**, 770-784.

[21] Xin, X., L. Guo, and C.B. Feng (1996). Reduced-order controllers for continuous and discrete time singular H^∞ control problems based on LMI, *Automatica*, **32**, 171-185.

[22] Yeh, H.H., J.L. Rawson, and S.S. Banda (1993). Robust control design with real-parameter uncertainties, *Proc of the 32nd IEEE CDC*, Chicago, Illinois, U.S.A., 3249-3256.

[23] Yung, C.F. (1999). H^∞ controller reduction for nonlinear systems, *Proc.of the 38th IEEE CDC*, Phoenix, Arizona, U.S.A., 3790-3795.

[24] Yung, C.F. (2000). Reduced-order H^∞ controller design-An algebraic Riccati equation approach, *Automatica*, **36**, 923-926.

[25] Yung, C.F., Y.P. Lin, and F.B. Yeh (1996). A family of nonlinear H^∞ output feedback controllers, *IEEE Trans. Automat. Control*, **41**, 232-236.

[26] Yung, C.F., J.L. Wu, and T.T. Lee (1998). H^∞ control for more general nonlinear systems, *IEEE Trans. Automat. Control*, **43**, 1724-1727.