行政院國家科學委員會專題研究計畫成果報告

從幾何觀點研究奇異系統 $H-infinity$ 控制器降階問題

計畫編號： NSC-90-2213-E-019-015
執行期限：90年8月01日至91年7月31日
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Abstract

State-space formulas to the reduced-order $\mathcal{H}_\infty$ controllers for descriptor systems are given. The approach taken is mainly based on the solutions of two generalized algebraic Riccati equations (GARE), while exploiting the structure of the deflating subspace of the pencil $\{\boldsymbol{E}, \boldsymbol{W}_\infty\}$, where $\boldsymbol{W}_\infty$ is an admissible solution to a GARE that is the descriptor systems counterpart of a certain ARE (algebraic Riccati equation) first developed by Petersen *et al.*(1991, *International Journal of Robust Nonlinear Control*, 1, 171-185). This approach has the advantage that, by proper selection of the bases of the deflating space and suitable assumptions, the reduced-order controller may be in a normal form, namely the $\boldsymbol{E}$-matrix of the controller is nonsingular.

1 Introduction

Since the success of the celebrating paper written by Doyle *et al*[3], much effort has been devoted to the control and design of the *generalized plant* for the $\mathcal{H}_\infty$ control problem. As suggested by the name "generalized plant," the to-be-controlled plant is usually not just the control system itself (usually it contains some weighting functions). Therefore the dimension of the plant can be very large. It is well known that the central controller has the same dimension as the plant, hence the controller might have relatively large dimension as well. One can easily recognize the need of the reduced-order controllers design in practical cases.

Recently, a number of papers have appeared that deal with reduced-order (or fixed-order) $\mathcal{H}_\infty$ controller design. The method of constrained optimization was first used in Hyland and Bernstein[9], and was further developed in Haddad and Bernstein[7], DeShelter and Ridgley[2], Sweriduk and Calise[17], and Iwasaki and Skelton[10]. An LMI technique and an algebraic Riccati inequality approach were used in Gahinet and Apkarian[5], and Juang *et al.*[11] and Hsu *et al.*[8], respectively. Most recently, Yung[21] has come up with an algebraic Riccati equation (ARE) approach in which the elementary idea was beginning with the bounded real lemma. Nevertheless, all the above mentioned literature tackled the conventional state-space systems, and less effort has been devoted to the reduced-order $\mathcal{H}_\infty$ control problem for descriptor systems. While the state-space representation will continue to be very important, there has been an increasing interest in working with the descriptor form representation. This is mainly because that the state variables introduced in state-space systems often do not provide a physical meaning[15][18]. In addition, state-space equations cannot represent algebraic restrictions between state variables. Discriptor form representation provides a suitable way to handle such problems. It has been proven in the literature that descriptor systems have higher capability in describing a physical system[15].

The solution to the $\mathcal{H}_\infty$ control problem for descriptor systems has lately been shown that it can be characterized by two generalized algebraic Riccati equations (GARE) [18][19], or equivalently by two linear matrix inequalitise (LMI)[14]. Moreover, a parameterization of all the stabilizing $\mathcal{H}_\infty$ controllers for descriptor systems has also been published[20]. We believe it is time to derive the reduced-order $\mathcal{H}_\infty$ controller for descriptor systems, for the necessary materials are coming out in the recent years. In this paper, we give a solution to the reduced-order $\mathcal{H}_\infty$ controller design problem for descriptor systems. The approach taken is mainly based on the GARE, while incoporating the idea of the bounded real lemma for descriptor systems. It is shown that the reduced-order controller has a close relationship with the deflating subspace of the pencil $\{\boldsymbol{E}, \boldsymbol{W}_\infty\}$. The matrix $\boldsymbol{W}_\infty$ is an admissible solution to a particular GARE (see below for details) which is the descriptor systems counterpart of the ARE result first found by Petersen *et al.*[16]. Another advantage of the present method is that, by proper selection of the deflating subspace, the reduced-order controller can be chosen to be in a *conventional state-space form, namely the* $\boldsymbol{E}$*-matrix* of the controller is nonsingular.

Before concluding this section, let us briefly review some basic notions concerning descriptor systems. Consider a descriptor system described by the state equations

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{x} \in I\!R^n$ is the state, $\boldsymbol{u} \in I\!R^m$ and $\boldsymbol{y} \in I\!R^p$ are the input and output signals respectively. $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are constant matrices with compatible dimensions and $\boldsymbol{E}$ is a square matrix of rank $r < n$. $\{\boldsymbol{E}, \boldsymbol{A}\}$ is assumed to be regular. It is well known that a descriptor system contains three different modes: finite dynamic modes, impulsive modes and nondynamic modes. For a detailed definition, see [1]. Briefly, let $q \triangleq \deg\det(s\boldsymbol{E} - \boldsymbol{A})$. Then $\{\boldsymbol{E}, \boldsymbol{A}\}$ has q finite dynamic modes, $r - q$ impulsive modes and $n - r$ nondynamic modes. Furthermore, if $r = q$, then there exist no impulsive modes and in this case the system is said to be impulse-free. $\{\boldsymbol{E}, \boldsymbol{A}\}$ is called stable if there exist no finite dynamic modes in $\mathrm{Re}[s] \geq 0$. $\{\boldsymbol{E}, \boldsymbol{A}\}$ is admissible if $\{\boldsymbol{E}, \boldsymbol{A}\}$ is regular, impulse-free and stable. The triple $\{\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{B}\}$ is said to be finite dynamics stabilizable and impulse controllable if there exists a constant matrix $\boldsymbol{K}$ such that $\{\boldsymbol{E}, \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\}$ is admissible. Similarly, $\{\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{C}\}$ *is called finite dynamics detectable and* impulse observable if there exists a constant matrix $\boldsymbol{L}$ such that $\{\boldsymbol{E}, \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C}\}$ is admissible. Without loss of generality, we can assume that the system (1) has a Weierstrass form[4]: $\boldsymbol{E} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{N} \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}$, and $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix}$, where $\boldsymbol{N}$ is a nilpotent matrix (that is, $\boldsymbol{N}^k = 0$ for some positive integer k).

2 Preliminaries

In this section, we will review some necessary materials including deflating spaces of matrix pencils and previous results on $\mathcal{H}_\infty$ control for descriptor systems. Let us first begin with the following definition.

Definition 1 *Let $s\boldsymbol{E} - \boldsymbol{A}$ be a regular pencil with $\boldsymbol{E}, \boldsymbol{A} \in I\!R^{n\times n}$. The linear space $\mathcal{V}$ is called a deflating subspace of $s\boldsymbol{E} - \boldsymbol{A}$ if $\dim(\boldsymbol{E}\mathcal{V} + \boldsymbol{A}\mathcal{V}) = \dim\mathcal{V}$.*

The following two theorems are taken from Lancaster and Rodman[12].

The rem 2 *Let $s\boldsymbol{E} - \boldsymbol{A}$ be a regular pencil with $\det(\lambda_0\boldsymbol{E} - \boldsymbol{A}) \neq 0$. Then a subspace $\mathcal{S}$ is invariant*

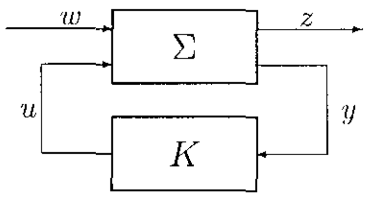

Figure 1: Standard Block Diagram

under $(\lambda_0 \boldsymbol{E} - \boldsymbol{A})^{-1}\boldsymbol{E}$ *if and only if* S *is deflating for the pair* $\{\boldsymbol{E}, \boldsymbol{A}\}$.

In the case that $\boldsymbol{E} = \boldsymbol{I}$ then S is $\boldsymbol{A}$-invariant, and so this concept generalizes that of an invariant subspace.

The rem 3 *Let* $\lambda \boldsymbol{E} - \boldsymbol{A}$ *be a regular matrix pencil. If* S *is an* r*-dimensional deflating subspace for* $\{\boldsymbol{E}, \boldsymbol{A}\}$ *then there are nonsingular matrices* $\boldsymbol{T}_1$ *and* $\boldsymbol{T}_2$ *for which*

$$\boldsymbol{T}_1\boldsymbol{E}\boldsymbol{T}_2 = \begin{bmatrix} \boldsymbol{E}_1 & \boldsymbol{E}_2 \\ 0 & \boldsymbol{E}_3 \end{bmatrix},$$
$$\boldsymbol{T}_1\boldsymbol{A}\boldsymbol{T}_2 = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ 0 & \boldsymbol{A}_3 \end{bmatrix}, \tag{2}$$

where $\boldsymbol{E}_1$, $\boldsymbol{A}_1$ *are of size* $r \times r$, *and* S *is the span of the first* r *columns of* $\boldsymbol{T}_2$. *Conversely, if* (2) *holds for nonsingular matrices* $\boldsymbol{T}_1$ *and* $\boldsymbol{T}_2$, *and* $\boldsymbol{V}_1$ *is the leading* $n \times r$ *partition of* $\boldsymbol{T}_2$ *then* $\Im m \boldsymbol{V}_1$ *is deflating for* $\{\boldsymbol{E}, \boldsymbol{A}\}$.

Consider the standard feedback configuration shown in Figure 1. Let the plant Σ be described by

$$\begin{aligned} \boldsymbol{E}\dot{x} &= \boldsymbol{A}x + \boldsymbol{B}_1\boldsymbol{w} + \boldsymbol{B}_2\boldsymbol{u} \\ &= \boldsymbol{C}_1 x + \boldsymbol{D}_{12}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}_2 x + \boldsymbol{D}_{21}\boldsymbol{w} \end{aligned} \tag{3}$$

where $\boldsymbol{x} \in I\!R^n$ is the state, and $\boldsymbol{w} \in I\!R^m$ represents a set of exogenous inputs which includes disturbances to be rejected and/or reference commands to be tracked. $\in I\!R^p$ is the output to be controlled and $\boldsymbol{y} \in I\!R^q$ is the measured output. $\boldsymbol{u} \in I\!R^l$ is the control input. $\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{D}_{12}$, and $\boldsymbol{D}_{21}$ are constant matrices with compatible dimensions. $\boldsymbol{E} \in I\!R^{n \times n}$ and rank$\boldsymbol{E} = r < n$.

The *standard* $\mathcal{H}_\infty$ *control problem for descriptor* systems consists of finding a controller $\boldsymbol{K}$ of the form

$$\begin{aligned} \hat{\boldsymbol{E}}\dot{\boldsymbol{\xi}} &= \hat{\boldsymbol{A}}\boldsymbol{\xi} + \hat{\boldsymbol{B}}\boldsymbol{y} \\ \boldsymbol{u} &= \hat{\boldsymbol{C}}\boldsymbol{\xi} \end{aligned} \tag{4}$$

where $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{A}} \in I\!R^{n \times n}$, $\hat{\boldsymbol{B}} \in I\!R^{n \times q}$ and $\hat{\boldsymbol{C}} \in I\!R^{l \times n}$, such that the resulting closed-loop system is internally stable and $\boldsymbol{T}_{\boldsymbol{zw}}$, the closed-loop system from $\boldsymbol{w}$ to , has $\mathcal{H}_\infty$ norm strictly less than a prescribed positive number γ. Here closed-loop internal stability means that the closed-loop system is regular and impulse-free, and that the states of Σ and $\boldsymbol{K}$ go to zero from all initial values when $\boldsymbol{w} = 0$. Note that we do not assume *a priori* structure of the matrix $\hat{\boldsymbol{E}}$; it may be singular or nonsingular, equal to $\boldsymbol{E}$ or not.

The system (3) is assumed to satisfy the following assumptions, see also[18].

(A1) $\{\boldsymbol{E}, \boldsymbol{A}\}$ is regular.
(A2) $\{\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{B}_2\}$ is finite dynamics stabilizable and impulse controllable.
(A3) $\{\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{C}_2\}$ is finite dynamics detectable and impulse observable.
(A4) $\begin{bmatrix} \boldsymbol{A} - \mathrm{j}\omega\boldsymbol{E} & \boldsymbol{B}_1 \\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} \end{bmatrix}$ has full row rank for all $\omega \in I\!R$ and is row reduced.
(A5) $\begin{bmatrix} \boldsymbol{A} - \mathrm{j}\omega\boldsymbol{E} & \boldsymbol{B}_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$ has full column rank for all $\omega \in I\!R$ and is column reduced.
(A6) $\boldsymbol{R}_1 \triangleq \boldsymbol{D}_{12}^T\boldsymbol{D}_{12} > 0$.
(A7) $\boldsymbol{R}_2 \triangleq \boldsymbol{D}_{21}\boldsymbol{D}_{21}^T > 0$.

The following theorem, which is taken from Wang *et al.*[19], is needed in our later development.

The rem 4 *Consider* (3). *Suppose that assumptions* (A1)-(A7) *hold. Then there exists a controller of the form* (4) *that internally stabilizes* (3) *and render* $\|\boldsymbol{T}_{zw}\|_\infty < \gamma$ *if and only if the following conditions are satisfied.*
(i) *There exists an admissible solution* $\boldsymbol{X}_\infty$ *to the GARE* $Ric_1(\boldsymbol{X}) = (\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{R}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1)^T\boldsymbol{X} + \boldsymbol{X}^T(\boldsymbol{A} - \boldsymbol{B}_2\boldsymbol{R}_1^{-1}\boldsymbol{D}_{12}^T\boldsymbol{C}_1) + \boldsymbol{C}_1^T(\boldsymbol{I} - \boldsymbol{D}_{12}\boldsymbol{R}_1^{-1}\boldsymbol{D}_{12}^T)\boldsymbol{C}_1 + \boldsymbol{X}^T(\frac{1}{\gamma^2}\boldsymbol{B}_1\boldsymbol{B}_1^T - \boldsymbol{B}_2\boldsymbol{R}_1^{-1}\boldsymbol{B}_2^T)\boldsymbol{X} = 0$, $\boldsymbol{E}^T\boldsymbol{X} = \boldsymbol{X}^T\boldsymbol{E}$, *with* $\boldsymbol{E}^T\boldsymbol{X}_\infty = \boldsymbol{X}_\infty^T\boldsymbol{E} \geq 0$.
(ii) *There exists an admissible solution* $_\infty$ *to the GARE* $Ric_2() = (\tilde{\boldsymbol{A}} - \boldsymbol{B}_1\boldsymbol{D}_{21}^T\boldsymbol{R}_2^{-1}\tilde{\boldsymbol{C}}_2) + {}^T(\boldsymbol{A} -$

$B_1 D_{21}^T R_2^{-1} \tilde{C}_2)^T - {}^T(\tilde{C}_2^T R_2^{-1} \tilde{C}_2 - \frac{1}{\gamma^2} F_\infty^T R_1 F_\infty) + \tilde{B}_1 \tilde{B}_1^T = 0$, $E = {}^T E^T$, *with* $E_\infty = {}_\infty^T E^T \geq 0$, *where* $\bar{A} = A + \frac{1}{\gamma^2} B_1 B_1^T X_\infty$, $\tilde{C}_2 = C_2 + \frac{1}{\gamma^2} D_{21} B_1^T X_\infty$, $\tilde{B}_1 = B_1(I - D_{21}^T R_2^{-1} D_{21})$. *When these conditions hold, one such controller is given by*

$$\begin{aligned}
\hat{E} &= E \\
\hat{A} = A_k &\triangleq A + B_2\hat{C} - \hat{B}C_2 + \\
&\quad \tfrac{1}{\gamma^2}(B_1 - \hat{B}D_{21})B_1^T X_\infty, \\
\hat{B} = B_k &\triangleq ({}_\infty^T C_2^T + (I + \tfrac{1}{\gamma^2} X_\infty {}_\infty)^T B_1 D_{21}^T) R_2^{-1}, \\
\hat{C} = C_k &\triangleq F_\infty = -R_1^{-1}(B_2^T X_\infty + D_{12}^T C_1).
\end{aligned} \tag{5}$$

3 Main Results

A reduced-order controller will be given in this section. The state-space formula of the reduced controller is based on the central controller given in Theorem 4. Henceforth, we suppose the conditions (i) and (ii) of Theorem 4 hold. In this case, according to Wang *et al.*[19], the following GARE

$$\begin{aligned}
Ric_3(W) &= A_0^T W + W^T A_0 + \frac{1}{\gamma^2} W^T B_0 B_0^T W \\
&\quad + C_0^T C_0 = 0, \\
W^T E &= E^T W
\end{aligned} \tag{6}$$

has also an admissible solution W_∞ with $W_\infty^T E = E^T W_\infty \geq 0$, where

$$\begin{aligned}
A_0 &= \bar{A} - B_1 D_{21}^T R_2^{-1} \tilde{C}_2 - {}_\infty^T \tilde{C}_2^T R_2^{-1} \tilde{C}_2 \\
B_0 &= \tilde{B}_1 - {}_\infty^T \tilde{C}_2^T R_2^{-1} D_{21} \\
C_0 &= -R_1^{-\frac{1}{2}}(B_2^T X_\infty + D_{21}^T C_1) = R_1^{\frac{1}{2}} F_\infty
\end{aligned}$$

Without loss of generality, assume that the pair $\{E, W_\infty\}$ has the following form.

$$\{E, W_\infty\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} W_\infty^{11} & 0 \\ W_\infty^{21} & W_\infty^{22} \end{bmatrix} \right\},$$

where the matrices W_∞^{11} and W_∞^{22} are symmetric. For technical reasons, we assume that $\{E, W_\infty\}$ is regular and impulse-free. Let $\Im m E^T W_\infty$ be an $\hat{r}$-dimensional vector space, i.e. W_∞^{11} is of rank $\hat{r}$. Then, we can choose a matrix $\hat{V}$ such that its columns form an orthonormal basis of $\Im m W_\infty^{11}$, and a matrix $\hat{U}$ such that its columns forms an orthonormal basis of $\mathrm{Ker} W_\infty^{11}$. Under these circumstances, it can be shown that

$$\begin{bmatrix} \hat{V}^T \\ \hat{U}^T \end{bmatrix} W_\infty^{11} \begin{bmatrix} \hat{V} & \hat{U} \end{bmatrix} = \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix}$$

where the matrix $\hat{W} = \hat{V}^T W_\infty^{11} \hat{V} \geq 0$. As a consequence of the orthogonality of the matrix $\begin{bmatrix} \hat{V} & \hat{U} \end{bmatrix}$, we have

$$W_\infty^{11} = \begin{bmatrix} \hat{V} & \hat{U} \end{bmatrix} \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{V}^T \\ \hat{U}^T \end{bmatrix} = \hat{V} \hat{W} \hat{V}^T$$

Next, in a similar way, let $\tilde{V}$ be an orthonormal basis matrix of $\Im m W_\infty^{22}$. Define the following two matrices

$$\begin{aligned}
V &= \begin{bmatrix} \begin{bmatrix} \hat{V} & \hat{U} \end{bmatrix} & 0 \\ 0 & \tilde{V} \end{bmatrix}, \\
Q &= \begin{bmatrix} I & 0 \\ -(W_\infty^{22})^{-1} W_\infty^{21} & I \end{bmatrix},
\end{aligned}$$

where the inverse of W_∞^{22} exists provided that $\{E, W_\infty\}$ is impulse-free. It is therefore not difficult to see

$$\begin{aligned}
V^T E Q V &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \\
V^T W_\infty Q V &= \begin{bmatrix} \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & \tilde{W} \end{bmatrix}.
\end{aligned}$$

Here the actual form of $\tilde{W}$ is not important. It is then straightforward to show that the following identity holds.

$$V_1^T W_\infty = \begin{bmatrix} \hat{W} & 0 \\ 0 & \tilde{W} \end{bmatrix} V_2 \triangleq W_2 V_2,$$

where

$$V_1 = \begin{bmatrix} \hat{V} & 0 \\ 0 & \tilde{V} \end{bmatrix}, \text{ and } V_2 = V_1^T Q^{-1}$$

Motivated by the work of Yung[21], we will now claim that $\Im m \begin{bmatrix} \hat{V} \\ 0 \end{bmatrix}$ is a deflating subspace for $\{E^T, \hat{A}_0^T\}$. Since the pair $\{E, A_0\}$ is regular and impulse-free, it is assumed without loss of generality that they have the following form

$$\{E, A_0\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \right\}. \quad (7)$$

Lemma 5 *Let the pair $\{E, W_\infty\}$ be regular and impulse-free. Then the equations*

$$E_{\hat{r}} V_1^T Q^{-1} = V_1^T E, \text{ and } F V_1^T Q^{-1} = V_1^T \hat{A}_0 \quad (8)$$

has a solution F, where $E_{\hat{r}} = \begin{bmatrix} I_{\hat{r}} & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. Let B_0 and C_0 be partitioned as follows

$$B_0 = \begin{bmatrix} B_{01} \\ B_{02} \end{bmatrix}, \; C_0 = \begin{bmatrix} C_{01} & C_{02} \end{bmatrix},$$

where the partitions are compatible with (7). Observe that GARE $Ric_3(W_\infty)$ can be written as

$$\begin{aligned} Ric_3(W_\infty) = {} & A_0^T W_\infty + W_\infty^T A_0 + \tfrac{1}{\gamma^2} W_\infty^T B_0 B_0^T W_\infty \\ & + C_0^T C_0 \\ = {} & \begin{bmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{bmatrix}, \end{aligned}$$

where the $(1,1)$-block is an ARE given by

$$\begin{aligned} & (A_1 + \frac{1}{\gamma^2} B_{01} B_{02}^T W_\infty^{21})^T W_\infty^{11} + \\ & W_\infty^{11}(A_1 + \frac{1}{\gamma^2} B_{01} B_{02}^T W_\infty^{21}) + \\ & C_{01}^T C_{01} + \frac{1}{\gamma^2}(W_\infty^{21})^T B_{02} B_{02}^T W_\infty^{21} + \\ & \frac{1}{\gamma^2} W_\infty^{11} B_{01} B_{01}^T W_\infty^{11} = 0. \end{aligned} \quad (9)$$

Let $v \in \mathrm{Ker} W_\infty^{11}$. Pre-multiplying (9) by v^T and post-multiplying by v yields

$$v^T C_{01}^T C_{01} v + \frac{1}{\gamma^2} v^T ((W_\infty^{21})^T B_{02} B_{02}^T W_\infty^{21}) v = 0. \quad (10)$$

Equation (10) implies that $C_{01} v = 0$ and $B_{02}^T W_\infty^{21} v = 0$. Therefore $\mathrm{Ker} W_\infty^{11} \subset \mathrm{Ker} C_{01}$ and $\mathrm{Ker} W_\infty^{11} \subset \mathrm{Ker} B_{02}^T W_\infty^{21}$. Next, post-multiply (9) by v to get

$$W_\infty^{11}(A_1 + \frac{1}{\gamma^2} B_{01} B_{02}^T W_\infty^{21}) v = W_\infty^{11} A_1 v = 0,$$

which in turn implies that $\mathrm{Ker} W_\infty^{11}$ is A_1-invariant; hence that $\Im m W_\infty^{11}$ is A_1^T-invariant or, put in other way, $\hat{V}$ is A_1^T-invariant. Because we can assume that $\{E, A_0\}$ was in the form (7), it is straightforward to see that the following identities hold.

$$\begin{aligned} V_1^T Q^T E^T V &= \begin{bmatrix} I_{\hat{r}} & 0 \\ 0 & 0 \end{bmatrix}, \\ V_1^T Q^T A_0^T V &= \begin{bmatrix} \hat{V}^T A_1 \hat{V} & * \\ 0 & I \end{bmatrix}, \end{aligned}$$

where the asterisk sign denotes irrelevent terms. The next thing to see is that $\mathrm{Ker} W_\infty \subset \mathrm{Ker} F_\infty$, which can be easily derived from GARE $Ric_3(W_\infty) = 0$; then, accordingly, there exists a matric H such that

$$H V_1^T Q^{-1} = F_\infty. \quad (11)$$

Because $\hat{A}_0 = (A_0 + B_2 F_\infty)$, it is then easy to see that

$$E_{\hat{r}} V_1^T Q^{-1} = V_1^T E, \text{ and } F V_1^T Q^{-1} = V_1^T \hat{A}_0$$

hold for some matrix F. This completes the proof. Q.E.D.

The following theorem which is the main result of this section characterizes the reduced-order controller by the algebraic equations stated above.

The rem 6 *Consider* (3). *Suppose that conditions* **(i)** *and* **(ii)** *of* Theorem 4 *hold. Suppose also that $\{E, W_\infty\}$ is regular and impulse-free. Let matrices V_1, F, H and W_2 be given as above and have compatible dimensions. Then the $n - r + \hat{r}$-th order controller*

$$\begin{aligned} E_{\hat{r}} \dot{\xi} &= F\xi + Gy, \\ u &= H\xi, \end{aligned} \quad (12)$$

is internally stabilizing for Σ and renders $\|T_{zw}\| < \gamma$, where in (12) *the matrices $E_{\hat{r}}$, F and H satisfy equation* (8) *and* (11), *and the matrix G satisfies*

$$G = V_1^T \hat{B}_0. \quad (13)$$

Proof. The proof of the theorem is essentially based on the Bounded Real Lemma(See [19]). Let us first write down the state equations of the closed-loop system (3)-(12) in the following.

$$\begin{aligned} E_c\dot{x}_c &= A_c x_c + B_c w, \\ &= C_c x_c, \end{aligned}$$

where

$$\begin{aligned} x_c &= col(x, \xi), \\ E_c &= \begin{bmatrix} E & 0 \\ 0 & E_{\hat{r}} \end{bmatrix}, \\ A_c &= \begin{bmatrix} A & B_2H \\ GC_2 & F \end{bmatrix}, \\ B_c &= \begin{bmatrix} B_1 \\ GD_{21} \end{bmatrix}, \\ C_c &= \begin{bmatrix} C_1 & D_{12}H \end{bmatrix}. \end{aligned}$$

Define an r.s.e.(restrict system equivalent) transformation pair given by

$$\{T_1, T_2\} = \left\{ \begin{bmatrix} I & 0 \\ V_1^T & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ -V_1^TQ^{-1} & I \end{bmatrix} \right\}.$$

With this transformation matrix in hand, the closed-loop system can now be rewritten as:

$$\begin{aligned} \hat{E}_c\dot{\hat{x}}_c &= \hat{A}_c\hat{x}_c + \hat{B}_c w, \\ &= \hat{C}_c\hat{x}_c, \end{aligned}$$

where $E_c = \hat{E}_c$, and

$$\begin{aligned} \hat{A}_c &\triangleq \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \\ &= \begin{bmatrix} A + B_2HV_1^TQ^{-1} & B_2H \\ M & F - V_1^TB_2H \end{bmatrix}, \\ \hat{B}_c &\triangleq \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \\ &= \begin{bmatrix} B_1 \\ -V_1^TB_1 + GD_{21} \end{bmatrix}, \\ \hat{C}_c &\triangleq \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 + D_{12}HV_1^TQ^{-1} & D_{12}H \end{bmatrix} \end{aligned}$$

with $M \triangleq -V_1^TA + GC_2 - V_1^TB_2HV_1^TQ^{-1} + FV_1^T$. Set

$$X_c \triangleq \begin{bmatrix} X_\infty & 0 \\ 0 & W_2 \end{bmatrix}.$$

Then, X_c satisfies $E_c^TX_c = X_c^TE_c \geq 0$. It can be shown that the matrix X_c thus constructed is an admissible solution to the following GARE.

$$\begin{aligned} Ric_S(X) &:= \hat{A}_c^TX_c + X_c^T\hat{A}_c + \hat{C}_c^T\hat{C}_c + \\ &\quad \frac{1}{\gamma^2}X_c^T\hat{B}_c\hat{B}_c^TX_c \\ &:= \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \\ &= 0, \\ E_c^TX_c &= X_c^TE_c \geq 0. \end{aligned}$$

To see this, observe first that S_{11} can be written as

$$S_{11} = \hat{A}_{11}^TX_\infty + X_\infty^T\hat{A}_{11} + \hat{C}_1^T\hat{C}_1 + \frac{1}{\gamma^2}X_\infty\hat{B}_1\hat{B}_1^TX_\infty^T.$$

By using (11), it is straightforward to show that $S_{11} = Ric_1(X_\infty) = 0$. Next, as a direct consequence of (8) and (13), we have

$$\begin{aligned} S_{12}^T &= W_2(-V_1^T\tilde{A} - V_1^TB_2F_\infty + G\tilde{C}_2 + \\ &\quad FV_1^TQ^{-1}) \\ &= 0. \end{aligned}$$

Finally, by some plain calculations, it is shown that

$$S_{22} = V_1^TQ^T \times Ric_3(W_\infty) \times QV_1 = 0.$$

With the above discussions, it is concluded that X_c is a solution to GARE $Ric_S(X_c) = 0$. It remains to claim that X_c is indeed admissible. Compute the following matrix

$$\hat{A}_c + \frac{1}{\gamma^2}\hat{B}_c\hat{B}_c^TX_c \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}.$$

Note that $H_{21} = 0$ by equations (8), (11) and (13). Therefore $\{E_c, \hat{A}_c + \frac{1}{\gamma^2}\hat{B}_c\hat{B}_c^TX_c\}$ is admissible if and only if $\{E, H_{11}\}$ and $\{E_{\hat{r}}, H_{22}\}$ are admissible. The matrix H_{11} has the following form

$$\begin{aligned} H_{11} &= (A - B_2R_1^{-1}D_{12}^TC_1) + \\ &\quad (\frac{1}{\gamma^2}B_1B_1^T - B_2R_1^{-1}B_2^T)X_\infty. \end{aligned}$$

In view of the above identity, we conclude that $\{E, H_{11}\}$ is admissible because X_∞ is an admissible solution to the GARE $Ric_1(X) = 0$. Now consider the matrix H_{22}. It can be shown by a routine calculation that

$$(\boldsymbol{A}_0 + \frac{1}{\gamma^2}\boldsymbol{B}_0\boldsymbol{B}_0^T\boldsymbol{W}_\infty)^T\boldsymbol{V}_1 = \boldsymbol{Q}^{-T}\boldsymbol{V}_1\boldsymbol{H}_{22}^T.$$

This together with the identity $\boldsymbol{E}_{\hat{r}}\boldsymbol{V}_1^T\boldsymbol{Q}^{-1} = \boldsymbol{V}_1^T\boldsymbol{E}$ yields

$$\begin{aligned}(s\boldsymbol{E} - (\boldsymbol{A}_0 + \frac{1}{\gamma^2}\boldsymbol{B}_0\boldsymbol{B}_0^T\boldsymbol{W}_\infty))^T\boldsymbol{V}_1 \\ = \boldsymbol{Q}^{-T}\boldsymbol{V}_1(s\boldsymbol{E}_{\hat{r}} - \boldsymbol{H}_{22})^T,\end{aligned}$$

which simply says that the set of the characteristic polynomials (finte and infinite) of the pencil $s\boldsymbol{E}_{\hat{r}} - \boldsymbol{H}_{22}$ is a subset of the pencil $s\boldsymbol{E} - (\boldsymbol{A}_0 + \frac{1}{\gamma^2}\boldsymbol{B}_0\boldsymbol{B}_0^T\boldsymbol{W}_\infty)$'s. Because the pair $\{\boldsymbol{E}, \boldsymbol{A}_0 + \frac{1}{\gamma^2}\boldsymbol{B}_0\boldsymbol{B}_0^T\boldsymbol{W}_\infty\}$ is admissibe, this has already claimed that $\{\boldsymbol{E}_{\hat{r}}, \boldsymbol{H}_{22}\}$ is also admissible.

So far it has been shown that $\boldsymbol{X}_c$ is an admissible solution to the GARE $Ric_S(\boldsymbol{X}_c) = 0$. By Bounded Real Lemma[19], it is concluded that the reduced-order controller (12) is indeed internally stabilizing for Σ and makes $\|\boldsymbol{T}_{zw}\| < \gamma$. This completes the proof. Q.E.D.

The algebraic method just proposed certainly provides a way to construct a controller in the conventional state-space form; namely, a controller of the following form

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= \boldsymbol{F}_n\boldsymbol{\xi} + \boldsymbol{G}_n\boldsymbol{y}, \\ \boldsymbol{u} &= \boldsymbol{H}_n\boldsymbol{\xi},\end{aligned} \tag{14}$$

Recall that $\boldsymbol{V}_n \triangleq [\hat{\boldsymbol{V}}^T \ 0]^T$ is an $\hat{r}$-dimensional deflating subspace of $\{\boldsymbol{E}^T, \hat{\boldsymbol{A}}_0^T\}$ with finite spectrum. Then the following identity

$$\hat{\boldsymbol{A}}_0^T\boldsymbol{V}_n = \boldsymbol{E}^T\boldsymbol{V}_n\boldsymbol{F}_n^T \tag{15}$$

admits a solution $\boldsymbol{F}_n$. Unfortunately, in the present situation, the assumption that $\{\boldsymbol{E}, \boldsymbol{W}_\infty\}$ is impulse-free is not fulfilled to guarantee the existence of the normal-form reduced-order controller. Right now let us assume that the following identity

$$\boldsymbol{H}_n\boldsymbol{V}_n^T\boldsymbol{E} = \boldsymbol{F}_\infty \tag{16}$$

has a solution $\boldsymbol{H}_n$. The following corollary describes a way to examine the existence of the normal controller of the form (14).

C r llary 7 *Consider* (3). *Suppose that conditions* **(i)** *and* **(ii)** *of* Theorem 4 *hold. Let matrices* $\boldsymbol{V}_n$, $\boldsymbol{F}_n$ *and* $\boldsymbol{H}_n$ *be given as above and have compatible dimensions. Then the reduced-order normal controller* (14) *is internally stabilizing for* Σ *and renders* $\|\boldsymbol{T}_{zw}\| < \gamma$, *where in* (14) *the matrices* $\boldsymbol{F}_n$ *and* $\boldsymbol{H}_n$ *satisfy equation* (15) *and* (16), *and the matrix* $\boldsymbol{G}_n$ *satisfies*

$$\boldsymbol{G}_n = \boldsymbol{V}_n^T\hat{\boldsymbol{B}}_0. \tag{17}$$

Acknowledgement

The work was supported by the National Science Council of the Republic of China under Grant NSC-90-2213-E-019-015.

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