A linguistic computation model for fuzzy evaluation system *

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Abstract

In situations such as multi criteria decision making or performance appraisal, linguistic values such as "good", "poor", or "fair" are usually used to represent results of evaluation. To grasp the vagueness of linguistic values, fuzzy numbers are introduced to represent uncertainties. Usually, input and output of the evaluation system for fuzzy environment are fuzzy numbers. However, if final results of the overall evaluation are represented by fuzzy numbers, it would be difficult to understand for ordinary people. In this paper, we are going to propose a linguistic computation model for performance evaluation, which takes linguistic values as input, computes on linguistic values and outputs results in terms of linguistic values.

1 Introduction

In occasions such as multi criteria decision making or performance appraisal, linguistic values such as "good", "poor", or "fair" are usually used to represent results of evaluation. To grasp the vagueness of linguistic values, fuzzy numbers are introduced to represent uncertainties. Usually, input and output of the evaluation system for fuzzy environment are fuzzy numbers. However, if final results of the overall evaluation are represented by fuzzy numbers, it would be difficult to understand for ordinary people. In this paper, we are going to propose a linguistic computation model for performance evaluation, which takes linguistic values as input, computes on linguistic values and outputs results in terms of linguistic values.

Let L be the set of linguistic values under consideration and F be the set of fuzzy numbers. Let be $\circ$ be an extended arithmetic operation on F. By an extended arithmetic operation, we mean an arithmetic operation on fuzzy numbers constructed from its corresponding arithmetic operation on crisp numbers by the extension principle. Given an injection (one-to-one) function f from L to F, we can define a arithmetic operation $\star$ on L corresponding to $\circ$. Let $S(\cdot)$ be a similarity measure on F. Let $f(L)$ be the image of L under f. In this paper, we show that operation $\star$ on L can be defined as in terms of $\circ$ on F. In other words, we can extend arithmetic operations such as add or multiplication to linguistic values and the linguistic computation model can thus be constructed.

2 Fuzzy numbers and arithmetic operations

Definition 2.1 *Given a fuzzy set A defined on X and any number $\alpha \in [0,1]$, the α-cut, A^α, and the strong α-cut, $A^{\alpha+}$, are the crisp sets*

$$A^\alpha = \{x \mid A(x) \geq \alpha\} \quad (1)$$

$$A^{\alpha+} = \{x \mid A(x) > \alpha\}. \quad (2)$$

That is, the α-cut (or the strong α-cut) of a fuzzy set A is the crisp set A^α (or the crisp set $A^{\alpha+}$) that contains all the elements of the universal set X whose membership grades in A are greater than or equal to (or only greater than) the specified value of α.

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If we convert each of the α-cuts A^α into a special fuzzy set A_α:

$$A_\alpha(x) = \alpha \cdot A^\alpha(x) \quad (3)$$

then an arbitrary fuzzy set A can be represented in terms of A_α.

Theorem 2.1 *[1] For any fuzzy set A*

$$A = \cup_{\alpha \in [0,1]} A_\alpha \quad (4)$$

Definition 2.2 *The support of a fuzzy set A within a universal set X is the crisp set that contains all the elements of X that have nonzero membership grades in A.*

Clearly, the support of A is exactly the same as the strong α-cut of A for $\alpha = 0$, i.e., A^{0+}.

Definition 2.3 *The height, $h(A)$, of a fuzzy set A is the largest membership grade obtained by any element in that set. Formally,*

$$h(A) = \sup_{x \in X} A(x) \quad (5)$$

Definition 2.4 *A fuzzy set A is called normal when $h(A) = 1$; it is called subnormal when $h(A) < 1$.*

Definition 2.5 *A fuzzy set A on real numbers is convex iff*

$$A(\lambda x_1 + (1-\lambda)x_2) \geq min[A(x_1), A(x_2)] \quad (6)$$

Definition 2.6 *A fuzzy set A on real numbers is qualified as a fuzzy number if it possess at least the following three properties:*

1. *A must be a normal fuzzy set;*
2. *A^α must be a closed interval for every $\alpha \in (0,1]$;*
3. *the support of A, $A^{\alpha+}$, must be bounded.*

Since α-cuts of any fuzzy number are required to be closed intervals for all $\alpha \in (0,1]$, every fuzzy number is a convex fuzzy set.

The following theorem shows that membership functions of fuzzy numbers may be, in general, piecewise-defined functions.

Theorem 2.2 *[1] A is a fuzzy number if and only if there exists a closed interval $[a,b] \neq \phi$ such that*

$$A(x) = \begin{cases} 1 & for\, x \in [a,b] \\ l(x) & for\, x \in (-\infty, a) \\ r(x) & for\, x \in (b, \infty), \end{cases} \quad (7)$$

where l is a function from $(-\infty, a)$ to $(0,1)$ that is monotonic increasing, continuous from the right, and such that $l(x) = 0$ for $x \in (-\infty, \omega_1)$; r is a function from (b,∞) to $[0,1]$ that is monotonic decreasing, continuous from the left, and such that $r(x) = 0$ for $x \in (\omega_2, \infty)$.

Fuzzy arithmetic is base on two properties of fuzzy numbers: (1) each fuzzy set , and thus also each fuzzy number, can fully and uniquely be represented by its α-cuts; and (2) α-cuts of each fuzzy number are closed intervals of real numbers for all $\alpha \in (0,1]$. These properties enable us to define arithmetic operations on fuzzy numbers in terms of arithmetic operations on their α-cuts.

Let $\star$ denote any of the four arithmetic operations on closed intervals: addition +, subtraction -, multiplication *, division /. Then

$$[a,b] \star [c,d] = \{f \star g \mid a \leq f \leq b, d \leq g \leq e\}. \quad (8)$$

That is the result of an arithmetic operation on closed intervals is again a closed interval.

The four arithmetic operations on closed intervals are defined as follows:

$$[a,b] + [d,e] = [a+d, b+e], \quad (9)$$

$$[a,b] - [d,e] = [a-e, b-d], \quad (10)$$

$$[a,b] * [d,e] = [min(ad, ae, bd, be), max(ad, ae, bd, be)], \quad (11)$$

and provided that $0 \in [d,e]$,

$$\begin{aligned}[a,b]/[d,e] &= [a,b] * [1/e, 1/d] \\ &= [min(a/d, a/e, b/d, b/e), max(a/d, a/e, b/d, b\end{aligned} \quad (12)$$

Let A and B denote fuzzy numbers and let $\star$ denote any of the four basic arithmetic operations. Then, we define a fuzzy set on real numbers, $A \star B$, by defining its α-cut, $(A \star B)^\alpha$, as

$$(A \star B)^\alpha = A^\alpha \star B^\alpha \quad (13)$$

for any $\alpha \in (0,1]$. Furthermore, $A \star B$ can be expressed as

$$A \star B = \cup_{\alpha \in [0,1]} (A \star B)_\alpha. \quad (14)$$

It can be shown that fuzzy numbers are closed under arithmetic operations.

Theorem 2.3 *[1] Let $\star \in \{+, -, *, /\}$, and let A, B denote continuous fuzzy numbers. Then, the fuzzy set $A \star B$ is also a continuous fuzzy number.*

3 Linguistic variables and linguistic values

The concept of a fuzzy number plays a fundamental role in formulating quantitative fuzzy variables. These are variables whose states are fuzzy numbers. when, in addition, the fuzzy numbers represent linguistic concepts, such as very small, small, medium, and so on, as interpreted in a particular context. the resulting constructs are usually called linguistic variables.

Each linguistic variable the states of which are expressed by linguistic terms interpreted as specific fuzzy numbers is defined in terms of a base variable, the values of which are real numbers within a specific range. A base variable is a variable in classical sense, exemplified by any phisical variable, (e.g., temperature, pressure, speed, voltage, humidity, etc.) as well other numerical variable, (e.g., age, interest rate, performance, salary, etc.) In a linguistic variable, linguistic terms (values) representing approximate values of a base variable, germane to a particular application, are captured by appropriate fuzzy numbers.

Each linguistic variable is fully characterized by quintuple (v, T, X, g, m) in which v is the name of variable, T is the set of linguistic values of v that refer to a base variable whose values range over a universal set X, g is a syntactic rule (a grammar) for generating linguistic values, m is a semantic rule that assigns to each linguistic value $t \in T$ its meaning, $m(t)$, which is a fuzzy set on X (i.e., $m : T \rightarrow \mathcal{F}(\mathcal{X})$).

Fuzzy logic is primarily concerned with quantifying and reasoning about vague or fuzzy terms that appear in our natural language. In fuzzy logic, these fuzzy terms are referred to as linguistic variables. Terms used in our natural language to describe some concept that usually has vague or fuzzy values. Examples of such terms are temperature, height, or speed. Typical values for term (linguistic variable) height would be short, medium, or tall. In the statement "Jack is young,", we are saying that the implied linguistic variable age has the linguistic value of young.

In fuzzy expert systems, we use linguistic variables in fuzzy rules. A fuzzy rule infers information about a linguistic variable contained in its conclusion from information about another variable contained in its premise. For example:

Rule 1

IF Speed is slow

THEN Make the acceleration high

Rule 2

IF Temperature is low

AND Pressure is medium

THEN Make the speed very slow

In fuzzy multi criteria decision making or fuzzy performance appraisal, linguistic variables correspond to criteria or performance indicators. Fuzzy measurements for criteria or performance indicators are linguistic values which are usually represented by fuzzy numbers. Arithmetic operations or aggregations such as sum or average are then applied to these fuzzy numbers to provide decisive information for decision maker. These information falls into two major categories:(1) ranking of alternatives (2) overall performance ratings. In the last case, the overall performance ratings are represented by fuzzy numbers obtained from arithmetic operations or aggregations, with which it is difficult for decision maker to interpret.

In the following section, we are going to develop arithmetic and aggregating operations for linguistic values based on arithmetic operations of fuzzy numbers.

4 Arithmetic operations of linguistic values

Linguistic values of linguistic variables are usually embodied by fuzzy numbers which have nice properties such as closed under arithmetic operations and easy to manipulate if fuzzy numbers are in special forms such as triangular or trapezoidal representations.

Here we assume each linguistic value is represented as a fuzzy number. Let L be the set of linguistic values under consideration and F be the set fuzzy numbers over the universe of discourse which is the range of possible values of a linguistic variable. Namely, there is an one-to-one function (injection) from L to F. Let it be denoted as f. Let $S(\cdot)$ be a similarity measure on F. Let $f(L)$ be the image of L under f. Then arithmetic operations on L can be defined as follows:

Definition 4.1 *Let $\circ$ be an arithmetic operation on F. Then its corresponding arithmetic operation $\star$ on L is defined as:*
For l_1, $l_2 \in L$,

$$l_1 \star l_2 = l_3 \in L,$$

where

$$l_3 = f^{-1}(a)$$

and a satisfies

$$a = \max_{e \in f(L)} S(f(l_1) \circ f(l_2), e). \quad (15)$$

To facilitate computation, we adopt a new similarity measure for trapezoidal fuzzy numbers [2]. Let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. We define similarity between A and B be

$$S(A, B) = 1 - \frac{||A - B||_{l_p}}{||U||} \cdot 4^{\frac{-1}{p}}, \quad (16)$$

where l_p metric is defined to be

$$||A - B||_{l_p} = (\sum_{i=1}^{4} (| a_i - b_i |)^p)^{\frac{1}{p}}, \quad (17)$$

, U is the universe of discourse and

$$||U|| = \max(U) - \min(U) \quad (18)$$

Obviously,

$$0 \leq S(A, B) \leq 1, \quad (19)$$

and

$$S(A, B) = S(B, A). \quad (20)$$

If $A = (a, a, a, a)$ and $B = (b, b, b, b)$ are both crisp numbers, then

$$S(A, B) = 1 - \frac{| a - b |}{||U||}. \quad (21)$$

That is, crisp cases are encompassed by our new similarity measure. Let $p = 1$. If

$$A = (1, 2, 3, 4),$$

$$B = (2, 3, 4, 5),$$

$$C = (4, 5, 6, 7),$$

similarities

$$\begin{aligned} S(A, B) &= 1 - \tfrac{||A-B||_{l_p}}{||U||} \cdot 4^{\frac{-1}{p}} \\ &= 1 - \tfrac{||A-B||_{l_1}}{||U||} \cdot 4^{\frac{-1}{1}} \\ &= 1 - \tfrac{|1-2|+|2-3|+|3-4|+|4-5|}{7-1} \cdot 4^{-1} \\ &= 1 - \tfrac{4}{6} \cdot 4^{-1} \\ &= \tfrac{5}{6} \end{aligned}$$

$$\begin{aligned} S(A, C) &= 1 - \tfrac{||A-C||_{l_p}}{||U||} \cdot 4^{\frac{-1}{p}} \\ &= 1 - \tfrac{||A-C||_{l_1}}{||U||} \cdot 4^{\frac{-1}{1}} \\ &= 1 - \tfrac{|1-4|+|2-5|+|3-6|+|4-7|}{7-1} \cdot 4^{-1} \\ &= 1 - \tfrac{12}{6} \cdot 4^{-1} \\ &= \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} S(B, C) &= 1 - \tfrac{||B-C||_{l_p}}{||U||} \cdot 4^{\frac{-1}{p}} \\ &= 1 - \tfrac{||B-C||_{l_1}}{||U||} \cdot 4^{\frac{-1}{1}} \\ &= 1 - \tfrac{|2-4|+|3-5|+|4-6|+|5-7|}{7-1} \cdot 4^{-1} \\ &= 1 - \tfrac{8}{6} \cdot 4^{-1} \\ &= \tfrac{2}{3} \end{aligned}$$

New similarity measure has two merits: It is simple to compute and similarity between two fuzzy numbers can be computed even if they are disjoint.

5 Conclusions

In this paper we have developed arithmetic operations on linguistic values based on arithmetic operations on fuzzy numbers. Based our development, aggregation functions for linguistic values can be defined and a computation model for linguistic values can be built so that outputs of the computation are linguistic values in stead of fuzzy numbers, which are in forms more suitable for decision makers.

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