Ranking Fuzzy Numbers with Extended Fuzzy Preference Relation *

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Abstract

The crux of this paper is to propose a new ranking method based on preference relation not only easy to compute but also perserving some good properties such as additivity. First, we extend the definition of fuzzy preference relaiton so that the degree of preference is not confined within 0 and 1. With the extension of the definition, we the propose an extended fuzzy preference relation satisfying additivity. By additivity, we mean if the degree of preference of fuzzy number A over fuzzy number B is x and B over C is y then A over C is $x+y$.

1 Introduction

Many methods for fuzzy ranking have been proposed. They can be classified into two categories. The first category is based on defuzzification. Various methods of defuzzification have been proposed. In the first category, fuzzy numbers are defuzzified into crisp numbers or the so-called utilities in some literatures. Then the ranking are done based on these crisp numbers. Though it is easy to compute, the main drawback of this type is that defuzzification tends to loss some information and thus is unable to grasp the sense of uncertainty. The other category is based on fuzzy preference relation. The advantage of this type is that uncertainties of fuzzy numbers are kept during ranking process. However, the fuzzy preference relations proposed thus far are too complex to compute. Yuan [10] has proposed criteria for measuring ranking method. Lee [7] has proposed a new fuzzy ranking method based on fuzzy preference relation satisfying all criteria proposed by Yuan.

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In this paper, We extend the definition of fuzzy preference relation and propose an extened fuzzy preference relation which satisfies additivity and is easy to compute. Ranking fuzzy numbers based our extened fuzzy preference relation would need only $O(n)$ computations of the preference relation which is faster than currently known methods that require $O(nlogn)$ computations.

2 Preliminaries

Definition 2.1 *The α-cut of fuzzy set A, A^α, is the crisp set $A^\alpha = \{x \mid \mu_A(x) \geq \alpha\}$. The support of A is the crisp set $Supp(A) = \{x \mid \mu_A(x) > 0\}$. A is normal iff $sup_{x\in U}\mu_A(x) = 1$, where U is the univeral set.*

Definition 2.2 *A fuzzy subset A of real number R is convex iff*

$$\mu_A(\lambda x+(1-\lambda)y) \geq (\mu_A(x)\wedge\mu_A(y)), \forall x, y \in R, \forall\lambda \in [0,1],$$

where $\wedge$ denotes the minimum operator.

Definition 2.3 *A is a fuzzy number iff A is a normal and convex fuzzy subset of R.*

Definition 2.4 *A triangular fuzzy number A is a fuzzy number with piecewise linear membership function μ_A defined by*

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & otherwise, \end{cases}$$

which can be denoted as a triplet (a_1, a_2, a_3).

Definition 2.5 *Let A and B be two fuzzy numbers. Let $\circ$ be a operation on real numbers, such as +, -, *, $\wedge$, $\vee$, etc. By extension principle, the extended*

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operation $\circ$ *on fuzzy numbers can be defined by*

$$\mu_{A\circ B}(z) = \sup_{x,y:z=x\circ y} \{\mu_A(x) \wedge \mu_B(y)\}. \quad (1)$$

Definition 2.6 *Let* A *be a fuzzy number. Then* A^L_α *and* A^U_α *are defined as*

$$A^L_\alpha = \inf_{\mu_A(z)\geq\alpha}(z) \quad (2)$$

and

$$A^U_\alpha = \sup_{\mu_A(z)\geq\alpha}(z) \quad (3)$$

respectively.

Definition 2.7 *A fuzzy preference relation* R *is a fuzzy subset of* $\Re \times \Re$ *with membership function* $\mu_R(A,B)$ *representing the degree of preference of fuzzy number* A *over fuzzy number* B.

1. *R is reciprocal iff* $\mu_R(A,B) = 1 - \mu_R(B,A)$ *for all fuzzy numbers* A *and* B.
2. *R is transitive iff* $\mu_R(A,B) \geq \frac{1}{2}$ *and* $\mu_R(B,C) \geq \frac{1}{2} \Rightarrow \mu_R(A,C) \geq \frac{1}{2}$ *for all fuzzy numbers* A, B *and* C.
3. *R is a fuzzy total ordering iff R is both reciprocal and transitive.*

If fuzzy numbers are compared based on fuzzy preference relations, then A *is said to be greater than* B *iff* $\mu_R(A,B) > \frac{1}{2}$.

Definition 2.8 *An extended fuzzy preference relation* R *is an extended fuzzy subset of* $\Re\times\Re$ *with membership function* $-\infty \leq \mu_R(A,B) \leq \infty$ *representing the degree of preference of fuzzy number* A *over fuzzy number* B.

1. *R is reciprocal iff* $\mu_R(A,B) = -\mu_R(B,A)$ *for all fuzzy numbers* A *and* B.
2. *R is transitive iff* $\mu_R(A,B) \geq 0$ *and* $\mu_R(B,C) \geq 0 \Rightarrow \mu_R(A,C) \geq 0$ *for all fuzzy numbers* A, B *and* C.
3. *R is additive iff* $\mu_R(A,C) = \mu_R(A,B) + \mu_R(B,C)$
4. *R is a total ordering iff R is both reciprocal, transitive and additive.*

If fuzzy numbers are compared based on extended fuzzy preference relations, then A *is said to be greater than* B *iff* $\mu_R(A,B) > 0$.

3 Ranking of fuzzy numbers

Various methods have been proposed for ranking fuzzy number since 1976 [1, 2, 3, 4, 5, 6, 8, 9, 10, 7]. In this paper, we concentrate on methods based on fuzzy preference relations. Here, we propose a new fuzzy preference relation. Before we dwell into our new fuzzy preference relation, we will review some selected methods based on fuzzy preference relations.

3.1 Nakamura's method

A well-know fuzzy preference relation P between fuzzy sets A_i and A_j is defined by Nakamura [8], that is,

$$\mu_P(A_i, A_j) = \begin{cases} \frac{d(A_{iL}, A_{iL}\wedge A_{jL}) + d(A_{iU}, A_{iU}\wedge A_{jU})}{d(A_{iL}, A_{jL}) + d(A_{iU}, A_{jU})} \\ \quad \text{for } A_i \neq A_j \\ 0.5 \\ \quad \text{for } A_i = A_j \end{cases} \quad (4)$$

Here $A_i \wedge A_j$ is the extended minimum of A_i and A_j. A_L and A_U are the lower boundary set of A and the upper boundary set of A, defined by $\mu_{A_L}(y) = sup_{y\geq x}\mu_A(x), \forall y \in R$, and $\mu_{A_U}(y) = sup_{y\leq x}\mu_A(x), \forall y \in R$, respectively, and $d(A_i, A_j) = \int_{x\in R} \mid \mu_{A_i}(x) - \mu_{A_j} \mid dx$, is the Hamming distance between A_i and A_j.
Nakamura's fuzzy preference relation is not robust and laks dicrimination in some special cases.

3.2 Yuan's method

Yuan [10] defined a fuzzy preference relation Q based on the extended difference $A_i \ominus A_j$. That is,

$$\mu_Q(A_i, A_j) = \begin{cases} \frac{\lambda_1+\lambda_2}{\lambda} & \lambda > 0 \\ 0.5 & \lambda = 0 \end{cases} \quad (5)$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$,

$$\lambda_1 = \int_{\alpha:(A_i\ominus A_j)^U_\alpha > 0} (A_i \ominus A_j)^U_\alpha d\alpha,$$

$$\lambda_2 = \int_{\alpha:(A_i\ominus A_j)^L_\alpha > 0} (A_i \ominus A_j)^L_\alpha d\alpha,$$

$$\lambda_3 = \int_{\alpha:(A_i\ominus A_j)^U_\alpha < 0} -(A_i \ominus A_j)^U_\alpha d\alpha,$$

$$\lambda_4 = \int_{\alpha:(A_i\ominus A_j)^L_\alpha < 0} -(A_i \ominus A_j)^L_\alpha d\alpha,$$

$$(A_i \ominus A_j)^U_\alpha = \sup_{\mu_{A_i\ominus A_j}(z)\geq\alpha} (z),$$

$$(A_i \ominus A_j)^L_\alpha = \inf_{\mu_{A_i \ominus A_j}(z) \geq \alpha}(z).$$

Yuan has proved that Q is reciprocal, transitive, and robust.

3.3 An extended fuzzy preference relation

Our extended fuzzy preference relation is defined as follows.

Definition 3.1 *For any fuzzy number A, B, extended fuzzy preference relation $F(A,B)$ is defined by the membership function*

$$\mu_F(A,B) = \int_0^1 ((A-B)^L_\alpha + (A-B)^U_\alpha)d\alpha \qquad (6)$$

Lemma 3.1 *F is reciprocal, i.e.,*

$$\mu_F(B,A) = -\mu_F(A,B). \qquad (7)$$

Proof: Since

$$\begin{aligned} & (A-B)^L_\alpha + (A-B)^U_\alpha \\ = \; & A^L_\alpha - B^U_\alpha + A^U_\alpha - B^L_\alpha \\ = \; & -(B^L_\alpha - A^U_\alpha + B^U_\alpha - A^L_\alpha) \\ = \; & -((B-A)^L_\alpha + (B-A)^U_\alpha), \end{aligned} \qquad (8)$$

we have

$$\mu_F(B,A) = \mu_F(A,B). \qquad (9)$$

□

Lemma 3.2 *F is additive, i.e.,*

$$\mu_F(A,B) + \mu_F(B,C) = \mu_F(A,C) \qquad (10)$$

Proof:

$$\begin{aligned} & \mu_F(A,B) + \mu_F(B,C) \\ = \; & \int_0^1 ((A-B)^L_\alpha + (A-B)^U_\alpha)d\alpha + \\ & \int_0^1 ((B-C)^L_\alpha + (B-C)^U_\alpha)d\alpha \\ = \; & \int_0^1 A^L_\alpha - B^U_\alpha + A^U_\alpha - B^L_\alpha + B^L_\alpha - C^U_\alpha + B^U_\alpha - C^L_\alpha d\alpha \\ = \; & \int_0^1 ((A-C)^L_\alpha + (A-C)^U_\alpha)d\alpha. \end{aligned} \qquad (11)$$

□

Lemma 3.3 *F is transitive, i.e.,*

$$\mu_F(A,B) \geq 0 \textit{ and } \mu_F(B,C) \geq 0 \Rightarrow \mu_F(A,C) \geq 0. \qquad (12)$$

Proof: By lemma 3.2, we have

$$\mu_F(A,C) = \mu_F(A,B) + \mu_F(B,C).$$

Since

$$\mu_F(A,B), \mu_F(B,C) \geq 0,$$

we have

$$\mu_F(A,c) \geq 0.$$

□

Lemma 3.4 *Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. $\mu_F(A,B) \geq 0$ iff*

$$a_1 + 2a_2 + a_3 - b_1 - 2b_2 - b_3 \geq 0 \qquad (13)$$

Proof: $\mu_F(A,B) \geq 0$ iff

$$\begin{aligned} & \mu_F(A,B) \\ = \; & \int_0^1 (A-B)^L_\alpha + (A-B)^U_\alpha d\alpha \\ = \; & \frac{a_1 + 2a_2 + a_3 - b_1 - 2b_2 - b_3}{2} \geq 0. \end{aligned} \qquad (14)$$

□

Definition 3.2 *Let $\geq$ be a binary relation on fuzzy numbers defined by*

$$A \geq B \textit{ iff } \mu_F(A,B) \geq 0. \qquad (15)$$

Theorem 3.1 *$\geq$ is a total ordering relation.*

According to lemma 3.4, we have following lemma.

Lemma 3.5 *Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then*

$$A \geq B \textit{ iff } a_1 + 2a_2 + a_3 - b_1 - 2b_2 - b_3 \geq 0. \qquad (16)$$

Lemma 3.6 *Let $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ be a fuzzy number with parabolic membership function defined as*

$$\mu_A(x) = \begin{cases} \frac{-a_2}{2a_1} + \sqrt{\frac{a_2}{2a_1}^2 + \frac{(x-a_3)}{a_1}} & a_3 \leq x \leq a_1 + a_2 + a_3, \\ \frac{-a_5}{2a_4} - \sqrt{\frac{a_5}{2a_4}^2 + \frac{(x-a_6)}{a_4}} & a_1 + a_2 + a_3 \leq x \leq a_6, \\ 0, & \textit{otherwise}, \end{cases} \qquad (17)$$

Let $B = (b_1, b_2, b_3, b_4, b_5, b_6)$ be another fuzzy number with parabolic membership function. Let

$$Q(A,B) = \frac{1}{3}(a_1 + a_4 - b_1 - b_4) + \frac{1}{2}(a_2 + a_5 - b_2 - b_5) + (a_3 + a_6 - b_3 - b_6) \tag{18}$$

Then $A \geq B$ iff $Q(A,B) \geq 0$.

In the case of ranking more than two fuzzy numbers, $A_1, A_2, \ldots, A_n$, we may use the relation $F(A_i, A_j)$ for pairwise comparison and we need to calculate $(1/2)n(n-1)$ F values. To improve computational efficiency, we suggest to compare each fuzzy number A_i, $i = 1, 2, \ldots, n$, with the averge fuzzy number $\overline{A} = \sum_{i=1}^{n} A_i/n$. Then rank A_i according to $\mu_F(A_i, \overline{A})$.

Theorem 3.2 *It follows that the ranking method based on extended fuzzy preference relaiton F needs $O(n)$ computations of F, which is more efficient than any known method.*

4 Conclusions

In this paper, we have extended the definition of fuzzy preference relation and proposed an extended fuzzy preference relation. Ranking fuzzy numbers based our extened fuzzy preference relation would only need $O(n)$ computations of the preference relation which is faster than currently known methods that require $O(nlogn)$ computations.

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