Aggregation of fuzzy opinions under group decision making environment *

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Abstract

The crux of this paper is to propose a new method for aggregating individual fuzzy opinions into an optimal group consensus. By optimality, we mean the sum of weighted dissimilarity among aggregated consensus and individual opinions is minimized. First, a new similarity measure for trapezoidal fuzzy numbers is proposed. Then, we propose an iterative algorithm for approximating the optimal consensus of expert opinions. Finally, the importance of each expert is taken into consideration in process of aggregation.

1 Introduction

In multi-criteria decision making problems under group decision environment, there arise situations of conflict and agreement among opinions of experts. Thus, finding a group consensus to represent a common opinion of the group is an important issue under group decision environment. The gist of this paper is to address this problem and establish a procedure to aggregate individual opinions into an optimal consensus. Since subjectivity, vagueness and imprecision enter into the assessments of experts, we will use fuzzy set theory to deal with the fuzziness of human judgement.

Several methods have been proposed for drawing consensus from opinions of experts. Methods in [4, 7, 8, 9, 10, 11] are based fuzzy preference relation. Ishikawa et al. [6] and Xu et al. [12] proposed that judgements of experts are represented by an interval-value and a group consensus judgement is derived from cumulative frequency distribution. Bardossy et al. [2] advocated that opinions of experts should be represented by fuzzy numbers. Recently, Hsu [5] proposed a method called similarity aggregation method (SAM) to aggregate individual opinions of experts.

There arose several problems in Hsu's work [5]. Basically, Hsu assumed that supports of fuzzy numbers should intersect. Otherwise, the degree of similarity is zero. If all the fuzzy numbers representing opinions are disjoint, the aggregation process will not work. No conclusion will be drawn from the opinions. Under this circumstance, Hsu suggested Delphi method can be employed to inspire experts to modify opinions into nondisjoint ones so that aggregation process can be continued. The other major problem of Hsu's work is that there is no way to tell whether the aggregation weights of individual opinions derived from SAM are optimal or not.

2 Preliminaries

Let $\tilde{R}_i = (a_i, b_i, c_i, d_i)$ be a positive trapezoidal fuzzy number representing i-th expert's subjective estimate of the rating to a given criterion and alternative. Let $\tilde{R} = F(\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n)$ be the consensus of opinions. How to construct such F to combine these estimated ratings $\tilde{R}_i$ $(i = 1, 2, \ldots, n)$ is an important issue.

To what degree should experts' opinions be reflected in the combined opinion? To answer this question, we give a criterion to determine how good an aggregated opinion is in next section. To facilitate computation, we propose a new similarity measure for trapezoidal fuzzy numbers. Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. We define similarity between $\tilde{A}$ and $\tilde{B}$ be

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$$S_p(\tilde{A},\tilde{B}) = 1 - q\|\tilde{A}-\tilde{B}\|_{l_p}^p, \qquad (1)$$

where $q = \frac{1}{4\|U\|^p}$,
l_p metric is defined to be

$$\|\tilde{A}-\tilde{B}\|_{l_p} = (\sum_{i=1}^{4}(\mid a_i - b_i \mid)^p)^{\frac{1}{p}}, \qquad (2)$$

, U is the universe of discourse and

$$\|U\| = \max(U) - \min(U) \qquad (3)$$

Obviously,

$$0 \le S_p(\tilde{A},\tilde{B}) \le 1, \qquad (4)$$

and

$$S_p(\tilde{A},\tilde{B}) = S_p(\tilde{B},\tilde{A}). \qquad (5)$$

3 Optimal aggregation method

Let $S_2(\tilde{A},\tilde{B})$ be the similarity between trapezoidal fuzzy number $\tilde{A}$ and $\tilde{B}$ as defined in previous section. Then the dissimilarity between $\tilde{A}$ and $\tilde{B}$ is defined by

$$c - S_2(\tilde{A},\tilde{B}), \qquad (6)$$

where c is a constant greater than 1;i.e. $c > 1$. The choice of c will influence the effect of aggregation, which shall be discussed later. An optimal aggregated opinion is to minimize the sum of weighted dissimilarity between aggregated opinion and each individual opinion.

Let $\tilde{R}$ be the aggregated opinion and $\tilde{R}_i (i = 1,\dots,n)$ be the individual opinions. To find the optimal consensus amounts to solving the following problem:

$$\min Z_{m,c}(W,\tilde{R}) = \sum_{i=1}^{n}(w_i)^m(c - S_2(\tilde{R}_i,\tilde{R})), \qquad (7)$$

where m is an integer greater than 1, c is a constant greater than 1,

$$W = (w_1, w_2, \dots, w_n)$$

and

$$\sum_{i=1}^{n} w_i = 1.$$

Problem (7) is an analytical problem which is quite similar to fuzzy c-means problem [13] and has the advantage that using differential calculus one can determine necessary conditions for local optima. The derivation of necessary conditions is similar to the result in [1] (p.67). For clarity, we will derive the conditions and devise an iterative algorithm to solve these conditions.

Theorem 3.1 *$(W,\tilde{R})$ may be global minimal for $Z_{m,c}$ only if*

$$\tilde{R} = \frac{1}{\sum_{i=1}^{n}(w_i)^m}\sum_{i=1}^{n}(w_i)^m\tilde{R}_i \qquad (8)$$

$$w_i = \frac{\left(\frac{1}{c - S_2(\tilde{R}_i,\tilde{R})}\right)^{\frac{1}{m-1}}}{\sum_{j=1}^{n}\left(\frac{1}{c - S_2(\tilde{R}_j,\tilde{R})}\right)^{\frac{1}{m-1}}} \qquad (9)$$

Proof: Omitted. □

The systems described by equations (8) and (9) cannot be solved analytically. There exists, however, algorithm approximating them iteratively. To show the algorithm will converge, the convergence theorem of Zangwill [14] will be applied. In order to proceed, we repeat without proof the results of Zangwill essential for our need.

Let $f : \mathcal{R}^n \to \mathcal{R}$ be real function with domain D_f and let

$$S^* = \{x^* \in D_f | f(x^*) < f(y), y \in B^0(x^*, r)\}$$

where

$$B^0(x^*, r) = \{x \in \mathcal{R}^n | \|x - x^*\| < r, \|\cdot\| \text{ any norm}\}$$

is an open ball of radius r centered at point x^*. We may refer to S^* as the solution set of the optimization problem

$$\min_{D_f}\{f(x)\} \qquad (10)$$

Let A be a point-to-point map from D_f to D_f;i.e., $A : D_f \to D_f$. We call $\{x^{(l)}\}$ an iterative sequence generated by A where

$$x^{(l+1)} = A(x^{(l)}) = \ldots = A^{(l)}(x^{(0)}), l = 1, 2, \ldots$$

Next we attach to A a descent function g.

Definition 3.1 *$g : D_f \to \mathcal{R}$ is a descent function for (A, S^*) if*

$$g \text{ is continuous on } D_f \qquad (11)$$

$$x^* \notin S^* \Rightarrow g(A(x^*)) < g(x^*) \qquad (12)$$

$$x^* \in S^* \Rightarrow g(A(x^*)) \le g(x^*) \qquad (13)$$

g monitors the progress of A at driving sequences $\{x^{(l)}\}$ towards a point in S^*; g may or may not be f itself. With these preliminaries, we the theorem of Zangwill.

Theorem 3.2 *[14] Regarding (10), if*

$$g \text{ is a descent function for } (A, S^*) \qquad (14)$$

$$A \text{ is continuous on } (D_f - S^*) \quad (15)$$

$$\{x^{(l)}\} \subset K \subseteq D_f \text{ and } K \text{ is compact} \quad (16)$$

then for every sequence $\{x^{(l)}\}$ *generated by A, either*

$$\{x^{(l)}\} \to x^* \in S^* \text{ as } l \to \infty \quad (17)$$

or the limit of every convergent subsequence $\{x^{(l_k)}\}$ *of* $\{x^{(l)}\}$ *is a point* $x^* \in S^*$, *for arbitrary* $x^{(0)} \in D_f$.

Now we are ready to show that equations (8) and (9) can be approximated iteratively and the iteration will converge.

Theorem 3.3 *Equations (8) and (9) can be approximated iteratively as follows:*

$$\tilde{R}^{(l)} = \frac{1}{\sum_{i=1}^{n}(w_i^{(l)})^m}\sum_{i=1}^{n}(w_i^{(l)})^m \tilde{R}_i \quad (18)$$

$$w_i^{(l+1)} = \frac{\left(\frac{1}{c - S_2(\tilde{R}_i, \tilde{R}^{(l)})}\right)^{\frac{1}{m-1}}}{\sum_{j=1}^{n}\left(\frac{1}{c - S_2(\tilde{R}_j, \tilde{R}^{(l)})}\right)^{\frac{1}{m-1}}} \quad (19)$$

That is, equations (18) and (19) will converge as l *approaches infinity.*

Proof: Omitted. □

Let us now comment on the roles and importance of constants m and c. m is called the exponential weight and it reduce the influence of "noise" when computing the central consensus in equation (8). m reduces the influence of small w_i (opinions further away from the consensus of opinions) compared to that of large w_i (opinions close to consensus $\tilde{R}$). The larger $m > 1$ the stronger is the influence. In regard to constant c, if c approaches to infinity, $w_i = \frac{1}{n}$. That is, the aggregation is the same as averaging. If c approaches to 1, the result of aggregation is equal to one of opinions. That is $w_i = 1$ and $w_j = 0$ for $j \neq i$.

Now, we are ready to present an iterative algorithm aggregate opinions satisfying equations (8) and (9). The procedure can be summarized by the following steps:

Algorithm A

1. Each expert $E_i (i = 1, 2, \ldots, n)$ constructs a positive trapezoidal fuzzy number $\tilde{R}_i = (a_i, b_i, c_i, d_i)$ to represent the subjective estimate of the rating to a given criterion and alternative.
2. Set initial aggregation weight $0 < w_i^{(0)} (i = 1, 2, \ldots, n) < 1$ and $\sum_{i=1}^{n} w_i^{(0)} = 1$. Each iteration in this algorithm will be labeled l, where $l = 0, 1, 2, \ldots$.
3. Calculate

$$\tilde{R}^{(l)} = \frac{1}{\sum_{i=1}^{n}(w_i^{(l)})^m}\sum_{i=1}^{n}(w_i^{(l)})^m \tilde{R}_i$$

4. Let $W^{(l)} = (w_1^{(l)}, w_2^{(l)}, \ldots, w_n^{(l)})$. Calculate $W^{(l+1)}$ as follows:

$$w_i^{(l+1)} = \frac{\left(\frac{1}{c - S_2(\tilde{R}_i, \tilde{R}^{(l)})}\right)^{\frac{1}{m-1}}}{\sum_{j=1}^{n}\left(\frac{1}{c - S_2(\tilde{R}_j, \tilde{R}^{(l)})}\right)^{\frac{1}{m-1}}}$$

5. If $\|W^{(l+1)} - W^{(l)}\| \leq \varepsilon$, stop; otherwise, set $l = l + 1$ and go to step 3.

In real world, group decision is heavily influenced by the degree of importance of participants. For example, the opinions of executives should be more reflected in the final conclusion of group decision. Therefore, we are now considering the relative importance weight of each expert. Let the degree of importance of ith expert be e_i and

$$\sum_{i=1}^{n} e_i = 1.$$

If algorithm A stops with

$$\|W^{(l+1)} - W^{(l)}\| \leq \varepsilon,$$

let $W^{(l+1)} = (w_1, w_2, \ldots, w_n)$. Then aggregation coefficient of ith expert is

$$AC_i = \beta \frac{(w_i)^m}{\sum_{i=1}^{n}(w_i)^m} + (1 - \beta)e_i, \quad (20)$$

where $0 \leq \beta \leq 1$. If $\beta = 1$, the degree of importance of experts is not considered in the aggregation process. If $\beta = 0$, only the degree of importance of experts is reflected in the consensus. Aggregation result $\tilde{R}$ can be defined as

$$\tilde{R} = \sum_{i=1}^{n}(AC_i \odot \tilde{R}_i). \quad (21)$$

Now we are ready to propose a revised method of algorithm A, called optimal aggregation method (OAM), which is summarized as follows:

Algorithm OAM

1. Each expert $E_i (i = 1, 2, \ldots, n)$ constructs a positive trapezoidal fuzzy number $\tilde{R}_i = (a_i, b_i, c_i, d_i)$ to represent the subjective estimate of the rating to a given criterion and alternative.
2. Set initial aggregation weight $0 < w_i^{(0)} (i = 1, 2, \ldots, n) < 1$ and $\sum_{i=1}^{n} w_i^{(0)} = 1$. Each iteration in this algorithm will be labeled l, where $l = 0, 1, 2, \ldots$.
3. Calculate
$$\tilde{R}^{(l)} = \frac{1}{\sum_{i=1}^{n} (w_i^{(l)})^m} \sum_{i=1}^{n} (w_i^{(l)})^m \tilde{R}_i$$
4. Let $W^{(l)} = (w_1^{(l)}, w_2^{(l)}, \ldots, w_n^{(l)})$. Calculate $W^{(l+1)}$ as follows:
$$w_i^{(l+1)} = \frac{\left(\frac{1}{c - S_2(\tilde{R}_i, \tilde{R}^{(l)})}\right)^{\frac{1}{m-1}}}{\sum_{j=1}^{n} \left(\frac{1}{c - S_2(\tilde{R}_j, \tilde{R}^{(l)})}\right)^{\frac{1}{m-1}}}$$
5. If $\|W^{(l+1)} - W^{(l)}\| > \varepsilon$, set $l = l + 1$ and go to step 3.
6. Let $W^{(l+1)} = (w_1, w_2, \ldots, w_n)$. Calculate aggregation coefficient AC_i $(i = 1, 2, \ldots, n)$ by
$$AC_i = \beta \frac{(w_i)^m}{\sum_{i=1}^{n} (w_i)^m} + (1 - \beta) e_i$$
7. Aggregate opinions of experts by
$$\tilde{R} = \sum_{i=1}^{n} (AC_i \odot \tilde{R}_i)$$

Our optimal aggregation method (OAM) preserve some important properties presented in other literatures [2, 5].

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